Hence,

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

6. Using (1.) and (2.) and the fact that $f^2 \in \mathcal{R}([a,b])$ whenever $f \in \mathcal{R}([a,b])$, the identity $4fg = (f+g)^2 - (f-g)^2$ implies $fg \in \mathcal{R}([a,b])$.

7. Fix an $\epsilon > 0$. Let $P = \{x_0, ..., x_n\}$ be a partition of $[a,b]$ such that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1}) < \epsilon.$$

Note that for any $x, y \in [x_{i-1}, x_i]$,

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)| \leq M_i(f) - m_i(f).$$

The right handside of the above equation does not depend on x and y. This implies $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$. Hence

$$U(|f|, P) - L(|f|, P) < \epsilon,$$

which shows that $|f| \in \mathcal{R}([a,b])$. Moreover, for any partition P,

$$U(f,P) \leq U(|f|,P) \ \Rightarrow\ \inf_{P'} U(f,P') \leq U(|f|,P) \ \Rightarrow\ \inf_{P'} U(f,P') \leq \inf_{P} U(|f|,P).$$

Both f and $|f|$ being Riemann integrable implies

$$\int_a^b f(x)\ dx = \inf_{P'} U(f,P') \leq \inf_{P} U(|f|,P) = \int_a^b |f(x)|\ dx.$$

□

6.2 Improper Integrals

In this section, we consider functions defined on (half) open intervals. f is possibly unbounded or discontinuous.

Definition 26. (Improper Integral on Finite Intervals) Let f be a function defined on $(a,b]$ (possibly unbounded) such that $f \in R([c,b])$ for all $c \in (a,b)$. The improper interal of f on $(a,b]$, denoted by $\int_a^b f(x)\ dx$, is defined to be

$$\int_a^b f(x)\ dx = \lim_{c \to a^+} \int_c^b f(x)\ dx,$$

provided the limit exists. If the limit exists then the imrpoper integral is said to be convergent. Otherwise it is said to be divergent.

A similar definition can be given for f defined on $[a, b)$. If f is defined on $[a, p) \cup (p, b]$, then the improper integral of f on $[a, b]$ is defined as

$$\int_a^b f(x)\ dx = \int_a^p f(x)\ dx + \int_p^b f(x)\ dx,$$

provided each improper integrals on $[a, p)$ and $(p, b]$ exist.

Definition 27. Let f be a function defined on (a, b) (possibly unbounded) such that $f \in R([c, b])$ for all $c, d \in (a, b)$, $c < d$. We say the improper integral of f on (a, b) converges if for some $c \in (a, b)$, the improper integrals $\int_a^c f(x)\ dx$ and $\int_c^b$ converge. In that case, we define

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

It can be shown that if $\int_a^b f(x)\ dx$ converge, then for any $d \in (a, b)$,

$$\int_a^b f(x)\ dx = \int_a^d f(x)\ dx + \int_d^b f(x)\ dx.$$

Example 16. Let $f(x) = 1/x$, $x \in (0, 1]$. f is continuous on $[c, 1]$ for all $0 < c < 1$. Hence $f \in R([c, 1])$ and

$$\int_c^1 \frac{1}{x}\ dx = \ln(c) \to -\infty, \text{ as } c \to 0^+.$$

Hence the improper integral of $1/x$ on $(0, 1]$ diverges.

Note that if $f(x) = x^{-r}$, $r < 1$, then the improper integral of f on $(0, 1]$ converges and is equal to $1/(1 - r)$.

Definition 28. (Improper Integral on unbounded intervals) Let f be a function defined on $[a, \infty)$ (possibly unbounded) such that $f \in R([a, c])$ for all $c \in (a, \infty)$. The improper interal of f on $[a, \infty)$, denoted by $\int_a^\infty f(x)\ dx$, is defined to be

$$\int_a^\infty f(x)\ dx = \lim_{c \to \infty} \int_a^c f(x)\ dx,$$

provided the limit exists. If the limit exists then the improper integral is said to be convergent. Otherwise it is said to be divergent.

The same definition can be applied to $(-\infty, b]$.

Definition 29. Let f be a function defined on $(-\infty, \infty)$ (possibly unbounded). If the improper integrals $\int_{-\infty}^c f(x)\ dx$ and $\int_c^\infty f(x)\ dx$ exist for some $c \in (-\infty, \infty)$, then we say the improper integral of f on $(-\infty, \infty)$ converges and is defined as

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^c f(x) + \int_c^\infty f(x)\ dx.$$

Otherwise, we say $\int_\infty^\infty f(x)\ dx$ diverges.

Chapter 7

Differentiation

7.1 Derivative of functions defined on $[a, b] \subset \mathbb{R}$

Let f be a function (real) defined on $[a, b] \subset \mathbb{R}$. For any $x \in [a, b]$, consider the function $\phi(t)$ defined on $(a, b) \setminus \{x\}$,

$$\phi(t) = \frac{f(t) - f(x)}{t - x}, \ t \in (a, b), t \neq x.$$

If $\lim_{t \to x} \phi(t)$ exists, define

$$f'(x) = \lim_{t \to x} \phi(t). \tag{7.1}$$

Thus, if $f'(x)$ is defined, then for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\phi(t) - f(x)| = \left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon, \text{ whenever } |t - x| < \delta, \ t \neq x.$$

Remark 20.

1. We associate to each function f a function f' whose domain is the set of points in $[a, b]$ at which the limit (7.1) exists. If f' is defined at x, then we say f is differentiable at x. If f' is defined for all $x \in A \subset [a, b]$, then we say f is differentiable on A.

2. It is possible to define the right-hand and left-hand derivatives of f at x. I.e. for all $x \in (a, b)$, define the left-hand and right-hand derivatives as

$$f'_-(x) = \lim_{t \to x^-} \phi(t), \ f'_+(x) = \lim_{t \to x^+} \phi(t).$$

   Then for all $x \in (a, b)$, $f'(x)$ is defined if and only if $f'(x) = f'_-(x) = f'_+(x)$. Note that when x is either a or b, then $f'(x)$ is a right-hand or left-hand derivative, repsectively.

3. If f is only defined on (a, b), then the definition of $f'(x)$, $x \in (a, b)$, is as in (7.1). But $f'(a)$ and $f'(b)$ are not defined in this case.

4. If $f'(x)$ is defined at x, then there exists $u(t)$ defined on (a,b) such that
$$f(t)-f(x)=(t-x)[f'(x)+u(t)], \text{ and } \lim_{t\to x} u(t)=0.$$
On the other hand, if there exists $u(t)$ defined on (a,b) such that
$$\lim_{t\to x} u(t)=0, \text{ and } f(t)-f(x)=(t-x)[a+u(t)],$$
for some $a\in\mathbb{R}$, then $f'(x)$ exists and $f'(x)=a$.

Lemma 1. *Suppose f is defined on $[a,b]$ and f is differentiable at x. Then f is continuous at x.*

Proof. We have
$$f(t)-f(x)=\frac{f(t)-f(x)}{t-x}(t-x)\to f'(x)\cdot 0=0, \text{ as } t\to x.$$
Hence f is continuous at x. □

Proposition 27. *Suppose f and g are functions defined on $[a,b]\in\mathbb{R}$ and that both are differentiable at $x\in[a,b]$. Let $c\in\mathbb{C}$. Then the following functions are differentiable at x.*

$$(cf)'(x) = cf'(x) \tag{7.2}$$
$$(f+g)'(x) = f'(x)+g'(x) \tag{7.3}$$
$$(f\cdot g)'(x) = f(x)g'(x)+f'(x)g(x) \tag{7.4}$$

If $g(x)\neq 0$, then
$$\left(\frac{1}{g}\right)'=-\frac{g'(x)}{g(x)^2} \tag{7.5}$$

Proof. (Sketch). Write $g(t)-g(x)=(t-x)(g'(x)+u(t)$ and $f(s)-f(y)=(s-y)[f'(y)+v(s)]$, such that
$$\lim_{t\to x} u(t)=0 \text{ and } \lim_{s\to y} v(s)=0.$$

1. $(cf(s)-cf(y)=c(f(s)-f(y))=c((s-y)[f'(y)+v(s)])=(s-y)\,[cf'(y)-cv(s)]$. Since $cv(s)\to 0$ as $s\to y$, we have
$$(cf(y))'=cf'(y)$$

2. Similarly,
$$(f+g)(t)-(f+g)(x)=(f(t)-f(x))+(g(t)-g(x))=(t-x)\,[f'(x)+g'(x)+v(t)+u(t)]$$

3.

$$\begin{aligned}((fg)(t)-(fg)(x)) &= f(t)g(t)-f(t)g(x)+f(t)g(x)-f(x)g(x)\\ &= f(t)(g(t)-g(x))+g(x)(f(t)-f(x))\\ &= (t-x)\left[(f(t)(g'(x)+u(t))+g(x)(f'(x)+v(t))\right].\end{aligned}$$

Hence

$$\begin{aligned}\frac{(fg)(t)-(fg)(x)}{t-x} &= f(t)g'(x)+g(x)f'(x)+(f(t)u(t)+g(x)v(t))\\ &\to f(x)g'(x)+g(x)f'(x),\end{aligned}$$

as $t\to x$.

4.

$$\begin{aligned}\frac{1}{g(t)}-\frac{1}{g(x)} = \frac{g(x)-g(t)}{g(x)g(t)} &= -\frac{1}{g(x)g(t)}(g(t)-g(x))\\ &= -\frac{1}{g(x)g(t)}\left[(t-x)(g'(x)+u(t))\right]\end{aligned}$$

□

Theorem 26. *(Chain Rule) Suppose g, a real-value function, and $f(g)$ are defined on $[a,b]$. (note that f is defined on the range of g). if $g'(x)$ and $f'(g(x))$ are defined at $x\in[a,b]$, then $(f(g))'(x)$ is defined at x with*

$$(f(g))'(x) = f'(g(x))g'(x).$$

In other words, $f(g)$ is differentiable at x.

Proof. Let $y=g(x)$. Write $g(t)-g(x)=(t-x)(g'(x)+u(t)$ and $f(s)-f(y)=(s-y)[f'(y)+v(s)]$, such that

$$\lim_{t\to x}u(t)=0 \text{ and } \lim_{s\to y}v(s)=0.$$

Then

$$\begin{aligned}h(t)-h(x) &= f(g(t))-f(g(x))=(g(t)-g(x))(f'(y)+v(g(t)))\\ &= (t-x)(g'(x)+u(t))(f'(y)+v(g(t))).\end{aligned}$$

Note that since g is continuous at x, we have $\lim_{t\to x}g(t)=g(x)=y$, and hence

$$\lim_{t\to x}v(g(t)) = \lim_{s=g(t)\to y}v(s)=0.$$

This implies

$$\begin{aligned}\lim_{t\to x}\frac{h(t)-h(x)}{t-x} &= \lim_{t\to x}[(g'(x)+u(t))(f'(y)+v(g(t))]\\ &= \left[\lim_{t\to x} g'(x)+u(t)\right]\left[\lim_{t\to x} f'(y)+v(g(t))\right]\\ &= g'(x)\cdot f'(y) = f'(g(x))g'(x).\end{aligned}$$

Therefore, $h'(x) = (f(g))'(x) = f'(g(x))g'(x)$. □

Example 17. Let f be defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0;\\ 0 & \text{if } x = 0.\end{cases}$$

Assume that $\sin'(x) = \cos(x)$. Then for $x \neq 0$, $f'(x) = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$. Now for $x = 0$, we have

$$\phi(t) = \frac{f(t)-f(0)}{t} = \sin\left(\frac{1}{t}\right),$$

which does not exists as $t \to 0$. Hence $f'(0)$ is not defined. However, if $a > 1$ and we define $f(x)$ as

$$f(x) = \begin{cases} x^a\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0;\\ 0 & \text{if } x = 0.\end{cases}$$

Then $f'(0) = 0$, and $f'(x) = ax^{a-1}\sin\left(\frac{1}{x}\right) - x^{a-2}\cos\left(\frac{1}{x}\right)$, for all $x \neq 0$. Note that for $a \in (1, 2]$, f' is not continuous at 0.

7.2 Mean Value Theorems

Definition 30. Let f be a real-value function defined on $D \subset \mathbb{R}$. We say f has a local maximum at a point $c \in D$ if there exists $\delta > 0$ such that $f(c) \geq f(t)$ for all $t \in N_\delta(c) \cap D$. The local minimum is defined in the same way but with $\leq$.

Theorem 27. *(Rolle's Theorem) Suppose f is defined on $[a, b] \subset \mathbb{R}$. If f has a local maximum at $c \in (a, b)$ and $f'(c)$ exists, then $f'(c) = 0$. The same result also holds for a local minimum.*

Proof. Let $\delta > 0$ such that $a < c - \delta < c + \delta < b$ and $f(c) \geq f(t)$ for all $t \in N_\delta(c)$.

if $t \in (c - \delta, c)$, then $f(t) \leq f(c)$ implies

$$\frac{f(t)-f(c)}{t-c} \geq 0.$$

Letting $t \to c$, we get $f'(c) \geq 0$. On the other hand if $t \in (c, c + \delta)$ then $f(t) \leq f(c)$ implies

$$\frac{f(t)-f(c)}{t-c} \leq 0.$$

Letting $t \to c$, we get $f'(c) \geq 0$. Thus, $f'(c) = 0$. □

Theorem 28. *(Generalized Mean Value Theorem) If f and g are continuous real-valued functions defined on $[a, b] \subset \mathbb{R}$, and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c). \tag{7.6}$$

Note that differentiability is not required at the endpoints.

Remark 21. If $g(b) - g(a) \neq 0$, then we have

$$\frac{m_f}{m_g} = \frac{f'(c)}{g'(c)}, \text{ for some } c \in (a, b).$$

Here $m_g = (f(b) - f(a))/(b - a)$ and $m_f = (f(b) - f(a))/(b - a)$. Let $g(x) = x$, then the above theorem implies there exists a $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Corollary 4. *(Mean Value Theorem) Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = (b - a)f'(c).$$

Proof. (Proof of the Generalized Mean Value Theorem). Let

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t).$$

Then $h(t)$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover, $h(a) = f(b)g(a) - g(b)f(a) = h(b)$.

Let $x, y \in [a, b]$ such that $h(x) = \inf_{t \in [a,b]} f(t)$ and $h(y) = \sup_{t \in [a,b]} h(t)$ (since h is continuous on $[a, b]$). If $h(x) = h(y)$, then $h = const$. Pick any $c \in (a, b)$, $h'(c) = 0$. Otherwise, $h(x) < h(y)$. Since $h(a) = h(b)$, then either $x \in (a, b)$ or $y \in (a, b)$. Let c be either x or y such that $c \in (a, b)$. We have $h'(c) = 0$ by Rolle's Theorem.

Thus, in all cases, there exists $c \in (a, b)$ such that $h'(c) = 0$, and hence

$$0 = h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c).$$

∎

Definition 31. A function f defined on (a, b) is said to be

1. monotone increasing if for all $a < x_1 < x_2 < b$, $f(x_1) \leq f(x_2)$.

2. strictly increasing if for all $a < x_1 < x_2 < b$, $f(x_1) < f(x_2)$.

3. monotone decreasing if for all $a < x_1 < x_2 < b$, $f(x_1) \geq f(x_2)$.

4. strictly decreasing if for all $a < x_1 < x_2 < b$, $f(x_1) > f(x_2)$.

Theorem 29. *Suppose f is differentiable on (a, b).*

i. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotone increasing.

ii. If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing.

iii. If $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant.

iv. if $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotone decreasing.

v. if $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing.

Proof. For any $a < x_1 < x_2 < b$, by the previous corollary, there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c).$$

Hence, if $f'(x) \geq 0$, for all $x \in (a, b)$, then

$$f(x_2) - f(x_1) \geq 0, \forall a < x_1 < x_2 < b,$$

which shows that f monotone increasing. Use the same idea to obtain $(ii) - (v)$. □

7.3 L'Hospital's Rule

L'Hospital's Rule is useful when it comes to evaluating

$$\lim_{t \to a} \frac{f(t)}{g(t)},$$

when $\lim_{t \to a} f(t) = 0 = \lim_{t \to a} g(t)$ or $\lim_{t \to a} f(t) = \pm\infty = \lim_{t \to a} g(t)$.

Theorem 30. *Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq \infty$. Moreover, suppose*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A \in [-\infty, \infty]. \tag{7.7}$$

If

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x), \tag{7.8}$$

or

$$\lim_{x \to a} g(x) = \infty \text{ as } x \to a, \tag{7.9}$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = A. \tag{7.10}$$

The analogous statement is also true if $x \to b$, or if $g(x) \to -\infty$.

Proof. Suppose $-\infty \le A < \infty$. Choose any $q \in \mathbb{R}$ such that $A < q$. Let $r = (q - A)/2$. (7.7) implies there exists $\delta > 0$ such that for all $x \in (a, a + \delta)$

$$\left| \frac{f'(x)}{g'(x)} - A \right| < (r - A) \;\Rightarrow\; \frac{f'(x)}{g'(x)} < r \tag{7.11}$$

Now if x and y are chosen such that $a < x < y < a + \delta$, then by the Generalized Mean Value Theorem, there exists $c \in (x, y) \subset (a, d)$ such that

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(c)}{g'(c)} < r. \tag{7.12}$$

Suppose (7.8) holds, then letting $x \to a$, we see from the above equation that

$$\frac{f(y)}{g(y)} \le r < q, \text{ for all } y \in (a, a + \delta). \tag{7.13}$$

Next, suppose (7.9) holds, then keeping y fixed in (7.13). we see that there exists $a < d_1 < y$ such that for all $a < x < d_1$, $g(x) > g(y)$ and $g(x) > 0$. Using (7.12), we have

$$\begin{aligned} \frac{f(x) - f(y)}{g(x)} &= \left[\frac{g(x) - g(y)}{g(x)} \right] \frac{f(x) - f(y)}{g(x) - g(y)} < \left[\frac{g(x) - g(y)}{g(x)} \right] r. \\ \Rightarrow \quad \frac{f(x)}{g(x)} &< \left[\frac{g(x) - g(y)}{g(x)} \right] r + \frac{f(y)}{g(x)} = r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}. \end{aligned}$$

Since $g(x) \to \infty$ as $x \to a$, the righthand side of the above equation approaches to r as $x \to a$. This implies that there exists $a < d_2 < d_1$ such that for all $a < x < d_2$,

$$\frac{f(x)}{g(x)} \le r + (q - r)/2 < q.$$

Thus, if either (7.8) or (7.9) holds, then

$$\frac{f(x)}{g(x)} < q, \; \forall x \in (a, d_2). \tag{7.14}$$

Using the same techniques, if $-\infty < A \le \infty$, and $p \in \mathbb{R}$ is chosen such that $p < A$, then there exists $d_3 > a$ such that

$$\frac{f(x)}{g(x)} > p, \; \forall x \in (a, d_3). \tag{7.15}$$

Combining (7.14) and (7.15), we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = A.$$

□