Math 300: Homework 5

Due Friday Feb. 26

1. Page 66: 3, 5, 6;

2. Let $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of disjoint open intervals in $\mathbb{R}$. I.e. $I_\alpha \cap I_\beta = \emptyset$ if $\alpha, \beta \in \mathcal{A}$ and $\alpha \neq \beta$. Show that $\mathcal{A}$ must be countable.

3. Prove that any open set in $\mathbb{R}$ can be written as a countable disjoint union of open intervals.

4. Find a subset A in $\mathbb{R}$ such that the followings are distinct.

$$A \quad int(A) \quad cl(int(A)) \quad int(cl(A))$$

5. Suppose A is a closed subset of $\mathbb{R}$ that is bounded above. Prove that the least upper bound of A belongs to A.

6. Let $A \subset \mathbb{R}$. Show that the closure of A is the smallest closed set containing A.

7. Let $X \subset \mathbb{R}$. Show that $O \subset X$ is relatively open in X if and only if $O = X \cap A$ for some A that is open in $\mathbb{R}$.

8. Let $X \subset \mathbb{R}$. Show that $F \subset X$ is relatively closed in X if and only if $F = X \cap B$ for some B that is closed in $\mathbb{R}$.

9. If $\{K_\alpha\}$ is a collection of compact subsets of $\mathbb{R}$ such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap_\alpha K_\alpha$ is nonempty. In particular, if $\{K_n\}$ is a sequence of nonempty closed and bounded sets in $\mathbb{R}$ such that $K_n \supset K_{n+1}$, for $n \geq 1$. Then $\cap_{n=1}^{\infty} K_n$ is not empty.

   Hint: Show by contradiction. Suppose that $\bigcap_\alpha K_\alpha = \emptyset$, then there exists a compact set K belonging to the collection such that each $x \in K$ does not belong to all of the other K_α's. Then $\{U_\alpha = (K_\alpha)^c : K_\alpha \neq K\}$ is an open cover for K. Use the fact that K is compact to find a contradiction.