Math 225: Homework 10

Due Thursday April 9

1. Find the characteristic polynomials, eigenvalues and eigenvectors of the following matrices:

   a.
   $$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

   b.
   $$\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

2. Find the characteristic polynomials, eigenvalues and eigenvectors of the following matrix:
   $$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

3. Let U be a vector space of dimension n. Assume that the characteristic polynomial of $A : U \to U$ has n distinct roots. Show that U has a basis consisting of eigenvectors of A.

4. Let A be an invertible matrix. If λ is an eigenvalue of A. show that $\lambda \neq 0$, and λ^{-1} is an eigenvalue of A^{-1}.

5. Let D denote the the derivative operator on differentiable functions. I.e. $D(f) = \frac{df}{dx}$. Show that $sin(kx)$ and $\cos(kx)$ are eigenvectors of D^2. What are the eigenvalues correspond to $\sin(kx)$ and $\cos(kx)$.

6. Let U be a vector space with a scalar product (positive definite). Let $A : U \to U$ be a symmetric linear map. We say A is positive definite if $(Av, v) > 0$ for all $v \in U$, $v \neq 0$. Prove:

   a. If A is positive definite, then all eigenvlaues of A are > 0.

   b. If A is positive definite, then there exists a symmetric linear map B such that $B^2 = A$, and $AB = BA$. Compute the eigenvalues of B.

7. Suppose A is a symmetric positive definite matrix (as in numerb 6). Show that A^2 and A^{-1} are positive definite symmetric matrices.

8. Let Let U be a vector space with an inner product. Let $A : U \to U$ be a Hermitian linear map. We say A is positive definite if $\langle Av, v\angle > 0$ for all $v \in U$, $v \neq 0$. Prove:

   a. If A is positive definite, then all eigenvlaues of A are > 0.

   b. If A is positive definite, then there exists a Hermitian linear map B such that $B^2 = A$, and $AB = BA$. Compute the eigenvalues of B.

9. An operator $A : U \to U$ is normal if $AA^* = A^*A$. Let A and B be normal matrices such that $AB = BA$. Prove that AB is normal. Also, state and prove the spectral theorem for normal matrices.

10. Find a unitary matrix P such that P^*AP is diagonal, where

$$\begin{pmatrix} 2 & 1+i \\ 1-i & 1 \end{pmatrix}$$