0. Notation

- Euclidean space $E_n (= \mathbb{R}^n)$:
  - $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, etc.
  - $x \cdot y := \sum_{j=1}^{n} x_j y_j$
  - $|x| := \sqrt{x \cdot x}$
- Function spaces:
  - For $1 \leq p < \infty$, $||f||_p := \left(\int |f(x)|^p dx \right)^{\frac{1}{p}}$, where $dx := dx_1 \cdots dx_n$ is the usual Lebesgue measure
  - $L^p(E_n)$ is the collection of measurable functions f for which $||f||_p < \infty$ for $1 \leq p < \infty$
  - $||f||_\infty := \text{esssup}\, |f|$; $L^\infty(E_n) := \{f \mid ||f||_\infty < \infty\}$
  - $C_0(E_n)$ is the set of all continuous functions vanishing at ∞.

1. The Schwartz Space

- **Multiindex notation.** If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers and $x \in \mathbb{R}^n$, then we make the following definitions

$$|\alpha| := \sum_{i=1}^{n} \alpha_i \qquad \alpha! := \prod_{i=1}^{n} (\alpha_i!)$$
$$x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i} \qquad \partial^\alpha := \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}}$$

- **Definition.** The Schwartz space $\mathcal{S}$ is the collection of all $\varphi \in C^\infty(E_n)$ such that for all pairs of multiindices α, β, one has

$$||\varphi||_{\alpha,\beta} := \sup_{x \in E_n} |x^\alpha \partial^\beta \varphi(x)| < \infty.$$

  Observe that $\mathcal{S}$ is a (complex) vector space.

- **Schwartz Space Topology.** The following two topologies on $\mathcal{S}$ are equal:
  1. (Fréchet Space Topology). Given any two multiindices α, β, any $\epsilon > 0$, and any $\varphi \in \mathcal{S}$, let $U_{\alpha,\beta,\varphi,\epsilon} := \{\psi \in \mathcal{S} \mid ||\varphi - \psi||_{\alpha,\beta} < \epsilon\}$. A set U is called open if it is a union (countable or uncountable) of finite intersections of sets $U_{\alpha,\beta,\varphi,\epsilon}$ (i.e., these sets are a sub-basis for the topology).
  2. (Metric Space Topolgy). For any $\varphi, \psi \in \mathcal{S}$, let

$$d(\varphi, \psi) := \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!} \min\{||\varphi - \psi||_{\alpha,\beta}, 1\}.$$

     This function d satisfies the definition of a metric (in particular, distance is always finite and satisfies the triangle inequality) and defines a metric topology.

  In either case, the following is true of convergence in $\mathcal{S}$: $\varphi_k \to \varphi$ iff $||\varphi_k - \varphi||_{\alpha,\beta} \to 0$ for all α, β.

- **Some Continuous Operations.** The following operations are continuous mappings of $\mathcal{S}$ to itself:

$$\begin{aligned}
\text{Differentiation: } & \varphi \mapsto \partial^\alpha \varphi \\
\text{Polynomial Multiplication: } & \varphi \mapsto p(x)\varphi(x) \text{ for any polynomial } p \\
\text{Translation: } & \varphi \mapsto \varphi(x - h) =: \tau_h \varphi(x) \\
\text{Modulation: } & \varphi \mapsto e^{2\pi i x \cdot \xi} \varphi(x) =: m_\xi \varphi(x) \\
\text{Lower Dilation: } & \varphi \mapsto \varphi(ax) =: \delta_a \varphi(x) \qquad (a > 0) \\
\text{Upper Dilation: } & \varphi \mapsto a^{-n} \varphi(a^{-1} x) =: \delta^a \varphi(x) \qquad (a > 0)
\end{aligned}$$

- **Corollary.** All functions in the Schwartz space are integrable in the Lebesgue sense and improperly Riemann-integrable. [Proof: The product $(1 + |x|^2)^n |\varphi(x)|$ is bouded above by some constant depending linearly on finitely many of the norms $||\varphi||_{\alpha,\beta}$.]

- **Other Important Facts.** The following facts are, for the most part, elementary but important properties of the Schwartz space (and reasonable practice for the interested reader).

  1. The C^∞ functions with compact support are dense in $\mathcal{S}$.
  2. Moreover, $\mathcal{S}$ has a countable dense subset.
  3. As a metric space, $\mathcal{S}$ is complete.
  4. Every $\varphi \in \mathcal{S}$ is uniformly continuous on E_n.
  5. For any $\varphi, \psi \in \mathcal{S}$, the pointwise product $\varphi\psi$ is also a Schwartz function (and mulitplication is a continuous operation).
  6. Given any $\varphi \in \mathcal{S}$, the translations $\tau_h \varphi$ converge to φ as $h \to 0$. Likewise with modulations and dilations.

     [Sketch of Proof: If $h = (h_1, \ldots, h_n)$, then the Fundamental Theorem of Calculus guarantees that

     $$\tau_h \varphi(x) - \varphi(x) = -\sum_{j=1}^n h_j \int_0^1 \frac{\partial \varphi}{\partial x_j}(x - \theta h) d\theta.$$

     Now differentiate by ∂^β and multiply by x^α. The binomial theorem guarantees that

     $$x^\alpha \partial^\beta (\tau_h \varphi(x) - \varphi(x)) = -\sum_{\gamma + \gamma' = \alpha} \sum_{j=1}^n \frac{\alpha!}{\gamma! \gamma'!} h_j h^{\gamma'} \int_0^1 \theta^{|\gamma'|} (x - \theta h)^\gamma \partial^\beta \frac{\partial \varphi}{\partial x_j}(x - \theta h) d\theta.$$

     Each integral over θ on the right-hand side is bounded by $\frac{1}{|\gamma'|+1} || \frac{\partial \varphi}{\partial x_j} ||_{\gamma', \beta}$ in magnitude, so the left-hand side must go to zero uniformly in x as $h \to 0$.]
  7. Difference quotients converge in $\mathcal{S}$: if e_j is the standard unit vector in the j-th coordinate direction, then for real h,

     $$\frac{\varphi - \tau_{\epsilon e_j} \varphi}{\epsilon} \to \frac{\partial \varphi}{\partial x_j}$$

     as $\epsilon \to 0$.

     [Sketch of Proof: As before,

     $$x^\alpha \partial^\beta \left(\frac{\varphi(x) - \tau_{\epsilon e_j} \varphi(x)}{\epsilon} \right) = \sum_{k=0}^{\alpha_j} \binom{\alpha_j}{k} \epsilon^k \int_0^1 \theta^k (x - \theta \epsilon e_j)^{\alpha - k e_j} \partial^\beta \frac{\partial \varphi}{\partial x_j}(x - \theta \epsilon e_j) d\theta.$$

     As $\epsilon \to 0$, the only nontrivial term on the right-hand side is $k = 0$. But the uniform convergence result just proved guarantees that

     $$\int_0^1 (x - \theta \epsilon e_j)^\alpha \partial^\beta \frac{\partial \varphi}{\partial x_j}(x - \theta \epsilon e_j) d\theta \to x^\alpha \partial^\beta \frac{\partial \varphi}{\partial x_j}(x)$$

     uniformly in x as $\epsilon \to 0$.]

- **Convolution.** Given two Schwartz functions φ, ψ, we define the convolution

$$\varphi \star \psi(x) := \int \varphi(y)\psi(x-y)dy.$$

(The integral can be taken in the Lebesgue sense or as an improper Riemann integral). Notice that $\varphi \star \psi = \psi \star \varphi$. In the same way, one can define the convolution of a Schwartz function with an integrable function (in particular, a continuous function with compact support).

- **Continuity of Convolution.** If φ is a Schwartz function or a continuous function with compact support and $\psi \in \mathcal{S}$ then $\varphi \star \psi$ is also Schwartz. Furthermore convolution is a continuous mapping from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{S}$.

  Proof: To clean notation, let $(p^\gamma \varphi)(x) := x^\gamma \varphi(x)$ be multiplication by the monomial x^γ. Now by induction, it is easy to establish the identity

$$x^\alpha \partial^\beta (\varphi \star \psi)(x) = \sum_{\gamma+\gamma'=\alpha} \frac{\alpha!}{\gamma!\gamma'!} (p^\gamma \varphi) \star (p^{\gamma'} \partial^\beta \psi)(x). \qquad (*)$$

  (**Note:** In particular, when differentiating a convolution, the derivative may be passed through to either function.) The norms $||\varphi \star \psi||_{\alpha,\beta}$ are finite because $p^\gamma \varphi$ is bounded and $p^{\gamma'} \partial^\beta \psi$ is integrable.

  Moreover, if $\varphi_k \to \varphi$ and $\psi_k \to \psi$ (both cases in $\mathcal{S}$), then $p^\gamma \varphi_k \to p^\gamma \varphi$ uniformly for each γ and $p^{\gamma'} \partial^\beta \psi_k \to p^{\gamma'} \partial^\beta \psi$ in $L^1(E_n)$, so the quantity $x^\alpha \partial^\beta (\varphi_k \star \psi_k)(x)$ tends uniformly to $x^\alpha \partial^\beta (\varphi \star \psi)(x)$.

- **One Last Fact.** If φ is fixed as some continuous function of compact support and $\psi \in \mathcal{S}$, then the RHS of $(*)$ is a uniform (in x) limit of Riemann sums (for the integral in the definition of convolution). In particular, then, if $\mathcal{P}$ stands for a partition of the support of φ into nonoverlapping rectangles R (with volume $|R|$, containing a point x_R)

$$\lim_{||\mathcal{P}|| \to 0} \sum_{R \in \mathcal{P}} \varphi(x_R)|R| \tau_{x_R} \psi = \varphi \star \psi$$

  in the Schwartz space topology (here $||\mathcal{P}||$ is the longest diameter of an $R \in \mathcal{P}$).

2. The Fourier Transform on $\mathcal{S}$

- **Definition.** Given $\varphi \in \mathcal{S}$, define

$$\hat{\varphi}(\xi) := \int e^{-2\pi i x\cdot\xi}\varphi(x)dx.$$

It is *a priori* clear that $\hat{\varphi}$ is defined at every point and a bounded function.

- **Riemann-Lebesgue Lemma.** For any $\varphi \in \mathcal{S}$, $||\hat{\varphi}||_\infty \le ||\varphi||_1$. Moreover, $\hat{\varphi}$ is continuous and vanishes at infinity.

[Proof: The norm inequality is obtained by bringing absolute values inside:

$$||\hat{\varphi}||_\infty \le \sup_{x\in E_n}\left|\int e^{-2\pi i x\cdot\xi}\varphi(x)dx\right| \le \sup_{x\in E_n}\int \left|e^{-2\pi i x\cdot\xi}\varphi(x)\right|dx = \int |\varphi(x)|dx = ||\varphi||_1.$$

Decay: Observe that $e^{-2\pi i x\cdot\xi} = e^{2\pi i h\cdot\xi}e^{-2\pi i(x+h)\cdot\xi}$; therefore, a change-of-variables gives that

$$(1-e^{-2\pi i h\cdot\xi})\hat{\varphi}(\xi) = \int e^{-2\pi i x\cdot\xi}\left(\varphi(x)-\tau_h\varphi(x)\right)dx$$

Now fix $h = \frac{\xi}{2|\xi|^2}$ to obtain that $|2\hat{\varphi}(\xi)| \le ||\varphi-\tau_h\varphi||_1$. As $\xi \to \infty$, the right-hand side tends to zero (because the difference tends to zero in the Schwartz topology and therefore in L^1 as well—because the integral is controlled by finitely many Schwartz norms).

Continuity: Now use the identity $e^{-2\pi i x\cdot\xi} = e^{2\pi i x\cdot h}e^{-2\pi i x\cdot(\xi+h)}$ in a similar fashion to obtain

$$\hat{\varphi}(\xi)-\hat{\varphi}(\xi+h) = \int e^{-2\pi i x\cdot\xi}(1-e^{-2\pi i x\cdot h})\varphi(x)dx.$$

Dominated convergence gives that the right-hand side goes to zero as $h \to 0$.]

- **Symmetries of the Fourier Transform.** Some of the symmetries of the Fourier transform include:

$$\begin{aligned} (\partial_j\varphi)^\wedge(\xi) &= 2\pi i\xi_j\hat{\varphi}(\xi) & (\tau_h\varphi)^\wedge &= m_{-h}\hat{\varphi} \\ (-2\pi i x_j\varphi)^\wedge &= \partial_j\hat{\varphi} & (m_h\varphi)^\wedge &= \tau_h\hat{\varphi} \\ (\delta_a\varphi)^\wedge &= \delta^a\hat{\varphi} & \overline{\hat{\varphi}(\xi)} &= (\overline{\varphi})^\wedge(-\xi) \end{aligned}$$

[Sketch of the Proof: Most of these identites follow from a simple change-of-variables. The multiplication/differentiation identies are a bit more subtle. For the first, observe that

$$m_{-\xi}\frac{\partial\varphi}{\partial x_j} = \lim_{\epsilon\to 0}\frac{m_{-\xi}\varphi - m_{-\xi}\tau_{\epsilon e_j}\varphi}{\epsilon}$$

in the Schwartz topology and therefore in the topology of L^1 as well. In particular, both sides can be integrated and the integral can pass through the limit. But

$$\int \frac{m_{-\xi}\varphi - m_{-\xi}\tau_{\epsilon e_j}\varphi}{\epsilon} = \int \frac{e^{-2\pi i x\cdot\xi} - e^{-2\pi i(x+\epsilon e_j)\cdot\xi}}{\epsilon}\varphi(x)dx$$

(to see this, leave the first term alone and make a change of variables $x \mapsto x + \epsilon e_j$ in the second term). The difference quotient on the right-hand side is bounded in magnitude by $|2\pi\xi_j|$ and converges pointwise to $2\pi i \xi_j e^{2\pi i \xi \cdot x}$ for all x as $\epsilon \to 0$. Dominated convergence gives the desired answer: $(\partial_j \varphi)^\wedge(\xi) = 2\pi i \xi_j \hat{\varphi}(\xi)$.

For the other identity,

$$\frac{\hat{\varphi}(\xi) - \tau_{\epsilon e_j}\hat{\varphi}(\xi)}{\xi} = \int \frac{e^{-2\pi i \xi \cdot x} - e^{-2\pi i (\xi - \epsilon e_j)\cdot x}}{\epsilon} \varphi(x) dx$$

This time, the difference quotient appearing on the right-hand side is bounded by $2\pi|x_j|$ for all x, ξ, ϵ and converges to $-2\pi i x_j e^{-2\pi i \xi \cdot x}$ as $\epsilon \to 0$. Now $|x_j \varphi(x)|$ is integrable, so dominated convergence gives that the right-hand side tends to the Fourier transform of $-2\pi i x_j \varphi(x)$ as $\epsilon \to 0$ (in particular, the partial derivatives of $\hat{\varphi}$ must exist everywhere).]

- **Continuity of the Fourier Transform.** For any multiindices α, β, $\xi^\alpha \partial^\beta \hat{\varphi}(\xi) = (-1)^{|\beta|}(2\pi i)^{|\beta|-|\alpha|}(\partial^\alpha(x^\beta \varphi))^\wedge(\xi)$—in particular, the Riemann-Lebesgue lemma guarantees that this is a continuous, bounded function for any α, β (so the Fourier transform at least maps $\mathcal{S}$ to itself). Now if $\varphi_k \to \varphi$ in the Schwartz topology, then it must also be the case that $||\partial^\alpha(x^\beta \varphi_k) - \partial^\alpha(x^\beta \varphi)||_1 \to 0$. The Riemann-Lebesgue lemma guarantees, then, that $\xi^\alpha \partial^\beta \hat{\varphi}_k(\xi)$ tends to $\xi^\alpha \partial^\beta \hat{\varphi}(\xi)$ uniformly in ξ as $k \to \infty$. Therefore the Fourier transform is continuous.

- **Some Easy Consequences of Fubini's Theorem**.

  1. Given $\varphi, \psi \in \mathcal{S}$,
  $$(\varphi \star \psi)^\wedge = \hat{\varphi}\hat{\psi}.$$
  2. **(Multiplication Formula).** Given $\varphi, \psi \in \mathcal{S}$,
  $$\int \hat{\varphi}(x)\psi(x)dx = \int \varphi(x)\hat{\psi}(x)dx.$$

- **One key computation:** Gaussians are in $\mathcal{S}$, and for any real $a > 0$,
$$\int e^{-\pi a|x|^2} e^{-2\pi i x \cdot \xi} dx = a^{-\frac{n}{2}} e^{-\pi|\xi|^2/a}.$$

[Sketch of Proof: By Fubini's theorem and the product structure of the identity, it suffices to prove it in the case $n = 1$. First complete the square:

$$\int e^{-\pi a x^2 - 2\pi i x \xi} dx = e^{-\pi \xi^2/a} \int e^{-\pi a (x + i\xi/a)^2} dx.$$

Now realize the right-hand side as a contour integral, and shift the contour (originally on the real axis) by an amount $-i\xi/a$. You're then left with integrating a good old-fashioned Gaussian with no complex phases.

An alternate route is to use the identity $(e^{-\pi a x^2})' = (-2\pi a x)e^{-\pi a x^2}$ to prove the Fourier transform of the Gaussian satsifes the ODE $y'(\xi) = (-2\pi\xi/a)y(\xi)$. All one needs to do is solve for $y(0)$ to completely determine y—this is again equivalent to integrating an ordinary Gaussian.]

3. Fourier Inversion on $\mathcal{S}$

- The goal is to prove the famous Fourier inversion formula: Given $\varphi \in \mathcal{S}$,

$$\varphi(x) = \int e^{2\pi i x\cdot\xi}\hat{\varphi}(\xi)d\xi. \tag{$*$}$$

- **Reduction.** First one must show that the pointwise identity

$$\int e^{2\pi i x\cdot\xi}\hat{\varphi}(\xi)d\xi = \lim_{a\to 0^+}(\delta^a\psi)\star\varphi(x)$$

holds, where $\psi(y) := e^{-\pi|y|^2}$.

[Proof: Let $\psi_{a,x}(y) := m_x e^{-\pi a|y|^2}$. By dominated convergence

$$\int e^{2\pi i x\cdot\xi}\hat{\varphi}(\xi)d\xi = \lim_{a\to 0^+}\int \psi_{a,x}(\xi)\hat{\varphi}(\xi)d\xi.$$

Now use the Multiplication formula to pass the Fourier transform to $\psi_{a,x}$. By the known symmetry properties of the Fourier transform, $\hat{\psi}_{\alpha,x}(y) = a^{-n/2}e^{-\pi|y-x|^2/a}$, so

$$\int e^{2\pi i x\cdot\xi}\hat{\varphi}(\xi)d\xi = \lim_{a\to 0^+}\int a^{-\frac{n}{2}}e^{-\pi|y-x|^2/a}\varphi(y)dy = \lim_{a\to 0^+}(\delta^{\sqrt{a}}\psi)\star\varphi(x).$$

Now $\sqrt{a}$ may be replaced by a without loss of generality.]

- **Definition.** A collection of nonnegative functions $\{\psi_a\}$ for $a > 0$ is called an approximate identity when the following properties hold:

  1. For each $a > 0$, $\int \psi_a(x)dx = 1$.
  2. For any $\delta > 0$, $\int_{|x|>\delta}\psi_a(x)dx \to 0$ as $a \to 0^+$.

  Observe that $\psi_a(x) := a^{-\frac{n}{2}}e^{-\pi|x|^2/a}$ (the upper dilates of the Gaussian appearing above) form an approximate identity.

- **Theorem.** If φ is a bounded, uniformly continuous function on E_n and $\{\psi_a\}$ is an approximate identity, then

$$\varphi(x) = \lim_{a\to 0^+}\psi_a\star\varphi(x)$$

with uniform convergence in x as $a \to 0^+$.

Proof. By property 1,

$$\varphi(x) - \psi_a\star\varphi(x) = \int[\varphi(x)-\varphi(x-y)]\psi_a(y)dy.$$

Pick any $\epsilon > 0$ and find the corresponding δ from the definition of uniform continuity. Suppose that $|\varphi(x)| \le M$; choose a sufficiently small by property 2 that its integral outside $[-\delta,\delta]$ is less than $\epsilon/2M$. Now

$$|\varphi(x) - \psi_a\star\varphi(x)| = \int_{|y|<\delta}|\varphi(x)-\varphi(x-y)|\psi_a(y)dy + \\ + \int_{|y|>\delta}|\varphi(x)-\varphi(x-y)|\psi_a(y)dy$$

(Note that there's no need for an absolute value on ψ_a since it's nonnegative). For the first integral, $|\varphi(x) - \varphi(x-y)| \leq \epsilon$ and $\int \psi_a(y)dy = 1$, so the first integral is bounded by ϵ. As for the second integral, $|\varphi(x) - \varphi(x-y)| \leq 2M$, but the integral of $\psi_a(x)$ on the set $|y| > \delta$ is less than $\epsilon/2M$, so all together $|\varphi(x) - \psi_a \star \varphi(x)| \leq 2\epsilon$.

- **Corollaries.**

  1. If $\varphi \in \mathcal{S}$ and $\hat{\varphi} = 0$ then $\varphi = 0$ (in particular, the Fourier transform is a homeomorphism of $\mathcal{S}$ to itself).
  2. **Fourier Inversion.** If $\varphi \in \mathcal{S}$, then $(*)$ holds.
  3. **Plancherel.** Given $\varphi \in \mathcal{S}$,

  $$\int |\varphi(x)|^2 dx = \int |\hat{\varphi}(\xi)|^2 d\xi.$$

  Proof: Let $\psi = \overline{\hat{\varphi}}$. The multiplication formula gives that

  $$\int |\hat{\varphi}(\xi)|^2 d\xi = \int \varphi(y)\hat{\psi}(y)dy.$$

  But $\hat{\psi}(y) = \overline{(\hat{\varphi})\hat{}(-y)} = \overline{\varphi(y)}$ by the Fourier inversion formula.

4. The Fourier Transform on $L^p(E_n)$, $1 \leq p \leq 2$.

- **The Fourier Transform on $L^1(E_n)$.** Given any $f \in L^1(E_n)$, its Fourier transform may be computed *a priori* by the same formula given for Schwartz functions:

$$\hat{f}(\xi) := \int e^{-2\pi i \xi \cdot x} f(x) dx.$$

  Using *exactly* the same proofs already given, the Riemann-Lebesgue lemma holds for $f \in L^1(E_n)$ (namely, that $||\hat{f}||_\infty \leq ||f||_1$ and $\hat{f}$ is continuous and vanishing at infinity). Likewise, the formulas

$$(\varphi \star \psi)^\wedge = \hat{\varphi}\hat{\psi} \text{ and } \int \hat{\varphi}\psi = \int \varphi\hat{\psi}$$

  hold when one merely assumes that $\varphi, \psi \in L^1(E_n)$ (note that Fubini's theorem guarantees that the convolution of two integrable functions exists almost everywhere and that $||\psi \star \phi||_1 \leq ||\psi||_1 ||\phi||_1$).

- **Density of $\mathcal{S}$ in $L^p(E_n)$ for $1 \leq p < \infty$.** The proof has two parts: first to show that for any $f \in L^p(E_n)$ and any $\epsilon > 0$, there exists $f_2 \in L^p(E_n)$ which is continuous and compactly supported such that $||f - f_2||_p < \epsilon/2$ (an important fact in its own right). The second part is to show that there exists a Schwartz function f_3 such that $||f_2 - f_3||_p < \epsilon/2$ (which will be accomplished by convolving f_2 with the approximate identity already encountered). By the triangle inequality, $||f - f_3||_p < \epsilon$.

  1. **Density of Continuous, Compactly-supported functions in $L^p(E_n)$ for $1 \leq p < \infty$:** Given $f \in L^p$ nonnegative and ϵ, there is a continuous function $\tilde{f}$, bounded with compact support, such that $||f - \tilde{f}||_p < \epsilon/3$.
     - Step 0: Given f, let $f = f_1 + if_2 - f_3 - if_4$ where f_1, f_2, f_3, f_4 are all nonnegative (simply restrict the real and imaginary parts of f to the sets where each is positive and negative; in this way it is also true that $||f_k||_p \leq ||f||_p$ for $k = 1, 2, 3, 4$). If $||f_k - g_k||_p < \epsilon/4$ for some continuous functions g_k with compact support, $k = 1, 2, 3, 4$, then the triangle inequality guarantees that $||f - (g_1 + ig_2 - g_3 - ig_4)||_p < \epsilon$. Therefore one may assume without loss of generality that $f \geq 0$.
     - Let A be some measurable set of finite measure, and by regularity $K \subset A \subset O$, with K compact, O open, and $|O \setminus K|$ arbitrarily small (with the closure of O compact as well). By Urysohn's lemma (for normal topological spaces), there is a continuous function φ with values in $[0, 1]$ such that $\varphi(x) = 1$ on K and $\varphi(x) = 0$ outside O. If you prefer, this function can be chosen explicitly to be equal to

$$\varphi(x) := \frac{\inf_{y \in O^c} |x - y|}{\inf_{y \in O^c} |x - y| + \inf_{z \in K} |x - z|}$$

       (the proof is left as an exercise). This function φ is close to χ_A in the topology of $L^p(E_n)$:

$$\int |\chi_A(x) - \varphi(x)|^p dx = \int_{O \setminus K} |\chi_A(x) - \varphi(x)|^p dx \leq 2^p |O \setminus K|.$$

– For any integers j, k, let $A_k^j := \{x \in E_n \mid j2^{-k} < f(x) \leq (j+1)2^{-k}\}$. Now

$$|A_k^j|(j2^{-k})^p = \int_{A_k^j} (j2^{-k})^p \leq \int_{A_k^j} |f|^p \leq ||f||_p^p$$

(this is known as the Tchebyshev Inequality). In particular, $|A_k^j| < \infty$. Now consider the function

$$f_k(x) := \sum_{j=1}^{\infty} j2^{-k} \chi_{A_k^j}(x).$$

By definition of A_k^j, $f_k(x) \leq f(x)$ for all x and all k. Moreover, $f_k(x)$ increases monotonically to $f(x)$ as $k \to \infty$. Therefore, for k and N sufficiently large, it must be the case that

$$\left|\left|\sum_{j=1}^{N} j2^{-k}\chi_{A_k^j} - f\right|\right|_p < \epsilon/6.$$

Approximating each function $\chi_{A_k^j}$ well enough by some continuous function φ_k^j of compact support (and multiplying by $j2^{-k}$ and summing), it follows that there must exist a continuous function $\tilde{f}$ of compact support such that $||f - \tilde{f}||_p < \epsilon/3$ as well.

2. **Approximation of continuous, compactly supported function by $\mathcal{S}$.** Suppose $f \in L^p(E_n)$ is continuous and compactly supported. Let M be chosen so that $|f(x)| \leq M$ for all x. For each $t > 0$, let

$$f_t(x) := \int t^{-\frac{n}{2}} e^{-\pi|x-y|^2/t} f(y)dy$$

(which exists because the integrand is continuous and compactly supported in y). Observe that this is a convolution of f with an approximate identity; it has already been shown that f_t is a Schwartz function for each $t > 0$, that $|f_t(x)| \leq M$, and that $f_t(x) \to f(x)$ uniformly as $t \to 0^+$. If one supposes that $f(y)$ is supported on the ball $|y| \leq R$ and x is any point with $|x| > 2R$, then

$$|x - y| \geq |x| - |y| \geq |x| - R \geq |x| - \frac{1}{2}|x|.$$

Consequently, one has that $e^{-\pi|x-y|^2/t} \leq e^{-\pi|x|^2/(4t)}$ when $|y| \leq R$ and $|x| > 2R$. Hence it follows that

$$|f_t(x)| \leq CMt^{-\frac{n}{2}} e^{-\pi|x|^2/(4t)}$$

for all $|x| > 2R$, where C is the volume of a ball of radius R. Now observe that

$$\int |f(x) - f_t(x)|^p dx \leq \int_{|x|\leq 2R} |f(x) - f_t(x)|^p dx + \int_{|x|>2R} (CMt^{-\frac{n}{2}} e^{-\pi|x|^2/(4t)})^p dx$$
$$\leq 2^n C||f - f_t||_\infty^p + (CMt^{-\frac{n}{2}})^p \int_{|x|>2R} e^{-p\pi|x|^2/(4t)} dx.$$

The first term goes to zero by uniform convergence of f_t to f. The second term goes to zero for purely computational reasons (show, for example, that the integral tends to zero faster than $e^{-c/t}$ for some constant c).

- **Consistency Check.** Suppose that φ_k is a sequence of Schwartz functions tending to some $f \in L^1(E_n)$ in the $L^1(E_n)$ topology. Since $(\varphi_j - \varphi_k)^\wedge = \hat{\varphi}_j - \hat{\varphi}_k$ and since φ_k is Cauchy in the L^1-distance, the Riemann-Lebesgue lemma implies that $\hat{\varphi}_k$ is Cauchy in the space $C_0(E_n)$ (because $||\hat{\varphi}_j - \hat{\varphi}_k||_\infty \leq ||\varphi_k - \varphi_j||_1$). What is the limit of $\hat{\varphi}_k$ in $C_0(E_n)$? Nothing more than $\hat{f}$, since

$$|\hat{f}(\xi) - \hat{\varphi}_k(\xi)| \leq \int |e^{-2\pi i \xi \cdot x}(f(x) - \varphi_k(x))| dx = ||f - \varphi_k||_1 \to 0$$

as $k \to \infty$.

- **The Fourier Transform on $L^2(E_n)$.** For a generic function $f \in L^2(E_n)$, the expression defining its Fourier transform makes no sense because f is not integrable. However, the Fourier transform has a sensible definition on $L^2(E_n)$ anyway. Let $\varphi_k \to f$ in $L^2(E_n)$ for some sequence of Schwartz functions φ_k. Just like the consistency check, Plancherel's theorem guarantees that $\hat{\varphi}_k$ converges in $L^2(E_n)$ becausethe sequence is Cauchy: $||\hat{\varphi}_j - \hat{\varphi}_k||_2 = ||\varphi_j - \varphi_k||_2$.

  - The Fourier Transform $\hat{f}$ of a function $f \in L^2(E_n)$ is *defined* to be the limit of the sequence $\hat{\varphi}_k$ in L^2 for any sequence of Schwartz functions φ_k converging to f in L^2.
  - If φ_k and ψ_k are two separate sequences of Schwartz functions converging to $f \in L^2$, then $||\hat{\varphi}_k - \hat{\psi}_k||_2 = ||\varphi_k - \psi_k||_2 \to 0$, so both sequences $\hat{\varphi}_k$ and $\hat{\psi}_k$ have the same limit (so the Fourier transform is well-defined).
  - The Plancherel formula still holds, that is, $||\hat{f}||_2 = ||f||_2$ for any $f \in L^2(E_n)$. This follows simply from the fact that the norm of the limit is the limit of the norms.
  - Likewise, simple limiting arguments show that the formulas

$$\varphi \star \psi = (\hat{\varphi}\hat{\psi})^\vee \text{ and } \int \hat{\varphi}\psi = \int \varphi\hat{\psi}$$

    hold when $\varphi, \psi \in L^2(E_n)$ (note that the convolution identity has changed form slightly from its previous incarnations because *a priori* $\varphi \star \psi$ is only a bounded function and so may not have a sensible Fourier transform, but $\hat{\varphi}\hat{\psi}$ is in $L^1(E_n)$).

- **The Fourier Transform on $L^p(E_n)$ for $1 \leq p \leq 2$.** Fix $\alpha > 0$. Given any $f \in L^p(E_n)$ for $1 \leq p \leq 2$, let E be the set on which $|f(x)| > \alpha$ and define $f_\infty(x) := f(x)\chi_E(x)$ and $f_0 := f - f_\infty$. Now $|f_0(x)| \leq \min\{|f(x)|, \alpha\}$ and $|f_\infty(x)| \leq \max\{|f(x)|, \alpha\}$, so $|f_0(x)|^2 \leq |f(x)|^p \alpha^{2-p}$ and $|f_\infty(x)| \leq |f(x)|^p \alpha^{1-p}$. It follows that $f_0 \in L^2(E_n)$ and $f_\infty \in L^1(E_n)$. Thus the Fourier transform of f can be defined as $\hat{f}_0$ (which defined almost everywhere as a function in $L^2(E_n)$) plus $\hat{f}_\infty$ (which is defined everywhere in terms of the usual integral).

  Suppose in general that $f = g + h = g' + h'$ where $g, g' \in L^1$ and $h, h' \in L^2$. It follows that $g - g' = h - h'$ and both functions are in both $L^1(E_n)$ and $L^2(E_n)$. The function

$g - g'$ may be approximated by a sequence of Schwartz functions φ_k which converges in both the L^1 and L^2 topologies (this is left as an exercise—step through the proof of density and show that the same sequences already produced are close to f in L^1 and L^2 if f is in both of these spaces). For this particular sequence φ_k, it must be the case that $\hat{\varphi}_k$ converges in $C_0(E_n)$ to the Fourier transform of $g - g'$ as defined in the L^1-sense. Furthermore, $\hat{\varphi}_k$ also converges in $L^2(E_n)$ to the Fourier transform of $g - g'$ as defined in the L^2-sense. It thus follows that the L^1 Fourier transform of $g - g'$ is square integrable and equal to the L^2 Fourier transform of $h - h'$ almost everywhere. Thus the Fourier transform of $f \in L^p$ as defined via splitting into $L^1 + L^2$ is independent of the particular splitting chosen.

- When $1 < p < 2$, there is a family of inequalities which bridges the Riemann-Lebesgue lemma and the Plancherel formula, referred to in general as the Hausdorff-Young inequality: $||\hat{f}||_{p'} \leq ||f||_p$ when $\frac{1}{p'} + \frac{1}{p} = 1$ and $1 \leq p \leq 2$. The Hausdorff-Young inequality follows from these two endpoint inequalities for completely abstract reasons. In fact, any time a linear operator is bounded from Banach spaces X_1 to Y_1 and X_2 to Y_2, there exist (non-trivial) Banach spaces X_θ and Y_θ for $\theta \in [0,1]$ such that the operator is bounded from X_θ and Y_θ. These ideas go by the name of interpolation theory. We will prove the Hausdorff-Yong inequality via the Riesz-Thorin interpolation theorem.

- When $p > 2$, there is no such procedure to define the Fourier transform (not merely due to a lack of ingenuity, but to impossibility). For this reason, the question of Fourier inversion is more subtle on L^p-spaces than it is for the Schwartz space. This question will be taken up shortly.

5. Tempered Distributions

- **Definition.** A tempered distribution F is a continuous linear functional on $\mathcal{S}$. The space of tempered distributions is denoted by $\mathcal{S}'$. Tempered distributions have a topology, called the weak topology: $F_k \to F \iff F_k(\varphi) \to F(\varphi)$ for all $\varphi \in \mathcal{S}$. The topology on $\mathcal{S}'$ will not play a significant role in this course.

- **Theorem.** F is a tempered distribution if and only if there exists a positive integer N and a constant $C < \infty$ such that

$$|F(\varphi)| \leq C \sum_{|\alpha|,|\beta| \leq N} ||\varphi||_{\alpha,\beta}.$$

Proof:

[$\Leftarrow$] Immediate from the convergence criterion for Schwartz norms.

[$\Rightarrow$] By definition of the topology on $\mathcal{S}$, any open set containing the origin in $\mathcal{S}$ must contain a set $U_{\epsilon,N}$ of the form

$$U_{\epsilon,N} := \left\{ \varphi \in \mathcal{S} \;\middle|\; \sum_{|\alpha|,|\beta| \leq N} ||\varphi||_{\alpha,\beta} < \epsilon \right\}.$$

Given any continuous linear functional F, $F^{-1}((-1,1))$ is open and contains the zero function. Thus, for some pair ϵ, N it must be the case that $|F(\varphi)| < 1$ on $U_{\epsilon,N}$. Let $||\varphi|| := \sum_{|\alpha|,|\beta| \leq N} ||\varphi||_{\alpha,\beta}$. Then for any $\varphi \neq 0$, one has $(\epsilon/2)\varphi/||\varphi|| \in U_{\epsilon,N}$, so $|F(\varphi)| \leq 2\epsilon^{-1}||\varphi||$.

- **Equality on Open Sets.** Distributions don't have any pointwise "values." However, the notion of equality on open subsets of E_n is well-defined. In particular, two tempered distributions F, G are said to be equal on the open set U if $F(\varphi) = G(\varphi)$ for all $\varphi \in \mathcal{S}$ which are supported in U. Given two distributions F and G, there is a unique, maximal open set U on which $F = G$ (the proof is an exercise; first restrict your attention to compactly supported Schwartz functions, then take a limit). The **support of a distribution** F is the unique minimal closed set E such that F equals the zero distribution on E^c.

- **Functions.** Many different classes of functions can be thought of as subsets of the class of tempered distributions.

  1. Suppose that f is an integrable function. Then f induces a distribution as follows:

  $$F_f(\varphi) := \int f(x)\varphi(x)dx.$$

  Notice that $|F_f(\varphi)| \leq ||f||_1 ||\varphi||_{0,0}$, so F_f is a tempered distribution. Typically, the distribution F_f will be identified with the function f.

  2. In the same way, functions $f \in L^p(E_n)$ for $1 < p \leq \infty$ induce distributions as well. By Hölder's inequality,

  $$\left| \int f(x)(\varphi(x) - \varphi_k(x))dx \right| \leq ||f||_p ||\varphi - \varphi_k||_{p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. It has already been shown that the L^1-norm and L^∞-norms of $\varphi - \varphi_k$ depend only on finitely many Schwartz norms of the differenc $\varphi - \varphi_k$. With these facts, one can show that the same is true for any $L^{p'}$-norm as well. Thus $F_f(\varphi_k) \to F_f(\varphi)$ when $\varphi_k \to \varphi$ in $\mathcal{S}$.

3. Most generally, any measurable function f which is locally integrable and satisfies
$$\int_{|x|<R} |f(x)| dx \leq CR^N$$
for some constants C, N and all $R > 0$ induces a tempered distribution via integration.

4. Two measurable functions f and g agree as distributions if and only if they are equal almost everywhere (the proof of this fact is postponed for now).

5. Any function with rapid growth at infinity (like e^x on the real line) does not induce a tempered distribution.

- **A Non-function.** The Dirac delta function is the tempered distribution defined by $F(\varphi) = \varphi(0)$. Notice that $|F(\varphi)| \leq ||\varphi||_{0,0}$. Furthermore, the distribution F is equal to zero away from the origin.

- **Extension of operators by duality.** Suppose that T is any mapping from some function space to some other function space, and suppose that there exists a continuous mapping T^* on $\mathcal{S}$ such that
$$\int Tf(x)\varphi(x)dx = \int f(x)(T^*\varphi)(x)dx$$
for all f and all $\varphi \in \mathcal{S}$. Then T can be extended to define a mapping on the space of tempered distributions by setting $(T(F))(\varphi) := F(T^*\varphi)$ for all $\varphi \in \mathcal{S}$. This definition always agrees with the old one (that is, if F is equal to the function f, then $T(F)$ is also equal to the function $T(f)$). Most every operation one can think of satisfies this property:
$$(\tau_h F)(\varphi) := F(\tau_{-h}\varphi) \qquad (\psi F)(\varphi) := F(\psi\varphi)$$
$$(m_\xi F)(\varphi) := F(m_\xi\varphi) \qquad (\delta_a F)(\varphi) := F(\delta^a\varphi)$$
$$(\psi \star F)(\varphi) := F(\widetilde{\psi} \star \varphi) \text{ where } \widetilde{\psi}(x) := \psi(-x)$$
$$(\partial^\beta F)(\varphi) := (-1)^{|\beta|} F(\partial^\beta\varphi) \text{ (integration-by-parts!)}$$

- **Fourier Transforms.** The Fourier transform is extended to a mapping $\mathcal{S}' \to \mathcal{S}'$ just like any other operator:
$$\hat{F}(\varphi) := F(\hat{\varphi}).$$
All of the symmetries are preserved (as can be easily proved):
$$(\tau_h F)^\wedge = m_{-h}\hat{F} \qquad \partial_k \hat{F} = -2\pi i (x_k F)^\wedge$$
$$(m_h F)^\wedge = \tau_h \hat{F} \qquad (\partial_k F)^\wedge = 2\pi i \xi_k \hat{F}$$
$$(\delta_a F)^\wedge = \delta^a \hat{F}$$

6. Tempered distributions, differentiation, and the Fourier transform

- Recall that a function $f \in L^p(E_n)$ is said to have weak partial derivatives in L^p if the difference quotients $\frac{1}{h}(f - \tau_{he_j} f)$ converge in L^p to some function g_j for $j = 1, \dots, n$ (where e_j is the j-th coordinate vector).

  Theorem. A function $f \in L^p(E_n)$ has weak partials in L^p if and only if the distributions $\frac{\partial f}{\partial x_j}$ are functions in $L^p(E_n)$ (and, in this case, $\frac{\partial f}{\partial x_j} = g_j$).

  First suppose that f has partial derivatives in the L^p-norm. Then

$$\int \frac{\partial f}{\partial x_j}\varphi = \lim_{h\to 0}\int \frac{1}{h}(f - \tau_{he_j} f)\varphi = \lim_{h\to 0}\int f\frac{1}{h}(\varphi - \tau_{-he_j}\varphi) = -\int f\frac{\partial \varphi}{\partial x_j}$$

  where the first equality holds, for example, by Hölder's inequality (since the convergence of functions is in $L^p(E_n)$ and $\varphi \in L^{p'}(E_n)$ for $\frac{1}{p} + \frac{1}{p'} = 1$) and the last holds because of the convergence of the difference quotients of φ in the Schwartz topology.

  In the reverse direction, the reasoning starts out the same way:

$$\int \frac{f - \tau_{he_j} f}{h}\varphi dx = \int f\frac{\varphi - \tau_{-he_j}\varphi}{h} dx = -\int f\int_0^1 \tau_{-\theta he_j}\frac{\partial\varphi}{\partial x_j} d\theta dx$$

  (the last equality follows directly from the Fundamental Theorem of Calculus in the usual way). Next, interchange the order of integration and pass the translation over to f by a change of variables:

$$\int \frac{f - \tau_{he_j} f}{h}\varphi dx = -\int_0^1 \tau_{\theta he_j} f\frac{\partial\varphi}{\partial x_j} dx d\theta.$$

  If f is assumed to have a weak partial derivative, the differentiation may be passed from φ to $\tau_{\theta he_j} f$ (observe that differentiation and translation commute when acting on distributions). Now changing the order of integration back gives

$$\int \frac{f - \tau_{he_j} f}{h}\varphi dx = \int \varphi\int_0^1 \tau_{\theta he_j}\frac{\partial f}{\partial x_j} d\theta dx. \qquad (*)$$

  One may then conclude that

$$\frac{f - \tau_{he_j} f}{h} = \int_0^1 \tau_{\theta he_j}\frac{\partial f}{\partial x_j} d\theta \qquad (**)$$

  almost everywhere, i.e., as functions in $L^p(E_n)$ (see the Minkowski integral inequality to guarantee that the RHS is an L^p function). This fact will be proved in full generality later; for now you may observe equality when $1 < p < \infty$ (pick any $g \in L^{p'}(E_n)$, approximate it by a sequence of Schwartz functions and use Hölder's inequality to show that $(*)$ still holds when φ is replaced by g and use the fact that L^p is the dual Banach space to $L^{p'}$). Assuming that the right-hand side tends to $\frac{\partial f}{\partial x_j}$ in the L^p-norm (which will be proved shortly), the theorem is complete.

- **Pointwise regularity.** Weak differentiability has connections to classical (i.e., pointwise) differentiability as well. To begin, we make a simplification. Suppose that $f \in L^p(E_n)$ has weak partial derivatives (of order up to M) in $L^p(E_n)$. Then for any smooth η with compact support, the function ηf is in $L^1(E_n)$ and has weak partial derivatives in $L^1(E_n)$ thanks to the fact that the Leibnitz rule is true for weak derivatives (the proof of which is a good exercise). Furthermore, if ηf is continuous, differentiable, etc., almost everywhere, then the same statement must also be true of f at every point in the interior of the support of η. Consequently, it suffices to consider weak derivatives of compactly supported L^1 functions.
  - **Convergence of Difference Quotients.** Suppose that $f \in L^p(E_n)$ has a weak j-th partial derivative in $L^p(E_n)$. Then after redefining f on a set of measure zero, the difference quotients converge pointwise almost everywhere (and, if $n = 1$, the function f is continuous).

    Proof: We may assume that $p = 1$ and that f is compactly supported. Fix h_0 sufficently *large* that the supports of f and $\tau_{h_0 e_j} f$ are disjoint. It follows from $(**)$ that on the support of f,

    $$f(x) = h_0 \int_0^1 \frac{\partial f}{\partial x_j}(x - \theta h_0 e_j) d\theta \text{ a.e.}$$

    (When $n = 1$, it can be shown that the right-hand side is continuous.) The advantage of redefining f by the expression on the right-hand side is that the equality $(**)$ now holds pointwise everywhere on the support of f for h sufficently small. Assuming that the right-hand side of $(**)$ tends to $\frac{\partial f}{\partial x_j}$ almost everywhere (which will be proved shortly alongside the convergence in the norm) it must be the case that this redefined f has convergent difference quotients.
  - **Example (Rademacher's Theorem).** A continuous function f on the real line is said to be Lipschitz if there exists a constant $C < \infty$ such that $|f(x) - f(y)| \leq C|x - y|$ for any two points x, y. Any Lipschitz function is differentiable almost everywhere, and the derivative is always bounded in magnitude by C.

    Proof: By the Tychonoff Product Theorem, there exists a sequence of real numbers $h_j \to 0$ such that the difference quotients

    $$\frac{f(x + h_j) - f(x)}{h_j}$$

    converge for each $x \in \mathbb{R}$ (because each difference quotient is constrained to have magnitude at most C at every point x). Let the limit function be called f'; as the limit of measurable functions, f' must be measurable. Now for any Schwartz function φ on the line,

    $$\int f'(x)\varphi(x) = \lim_{j \to \infty} \int \frac{f(x + h_j) - f(x)}{h_j} \varphi(x) dx =$$
    $$\lim_{j \to \infty} \int f(x) \frac{\varphi(x - h_j) - \varphi(x)}{h_j} dx = - \int f(x)\varphi'(x) dx$$

where the first inequality follows from dominated convergence and the last from convergence in the Schwartz topology. Since f is already continuous, the previous result shows that it must be differentiable almost everywhere.

  - **Continuity.** In general, one has to work a little bit harder for continuity in higher dimensions (because continuity along lines does not imply continuity in general). Here the theorem is as follows: Suppose that $f \in L^p(E_n)$ has weak derivatives $\partial^\alpha f$ which are also in $L^p(E_n)$ for all $|\alpha| \leq n+1$. Then f may be redefined on a set of measure zero such that it is continuous.

    Proof: Again, it suffices to assume that f and its weak derivatives are integrable and compactly supported. By Riemann-Lebesgue and the symmetries of the Fourier transform, $\xi^\alpha \hat{f}(\xi) \in C_0(E_n)$ for all α such that $|\alpha| \leq n+1$. In particular, there must exist some finite constant C such that $(1 + \sum_{i=1}^n |\xi_i|^{n+1})|\hat{f}(\xi)| \leq C$, which implies that $\hat{f}$ is integrable. Thus f, as a distribution, is equal to the inverse Fourier transform of $\hat{f}$, which is itself a function in $C_0(E_n)$. Thus the function f must be almost everywhere equal to a continuous function.

- **Compactly Supported Distributions.** In the case that a distribution F has compact support, there are a number of important facts that should be kept in mind:

  - The Fourier transform of a compactly supported distribution F is a real analytic function of at most polynomial growth.

    Proof: Fix some η which is C^∞ and compactly supported so that $\eta \equiv 1$ on the support of the distribution F. Now, for any Schwartz function ψ, $\psi - \eta\psi$ is supported away from the support of F; therefore $0 = F(\psi) - F(\eta\psi)$. Thus $\hat{F}(\varphi) = F(\hat{\varphi}) = F(\eta\hat{\varphi})$. For any real $\epsilon > 0$, consider the Schwartz function (depending on x):

$$\psi_\epsilon(x) := \epsilon^n \sum_{\xi \in \epsilon\mathbb{Z}^n} e^{-2\pi i x \cdot \xi} \eta(x) \varphi(\xi).$$

    As exercises, show that this sum converges in the Schwartz topology, and that $\psi_\epsilon \to \eta\hat{\varphi}$ in the Schwartz topology as $\epsilon \to 0^+$. By continuity of F, then, limits pass through and it follows that $\hat{F}(\varphi) = \int \varphi(\xi) F(m_{-\xi}\eta) dx$, or simply $\hat{F}(\xi) = F(m_{-\xi}\eta)$. This function is real analytic because the sum

$$e^{-2\pi i \xi \cdot x} \eta(x) = \sum_\alpha \frac{(-2\pi i)^{|\alpha|}}{\alpha!} \xi^\alpha x^\alpha \eta(x)$$

    converges uniformly in x (since η is compactly supported), as do all partial derivatives of this equation with respect to the x-variables; hence the sum converges in the Schwartz topology, and the limit passes through F to give that

$$\hat{F}(\xi) = \sum_\alpha \frac{(-2\pi i)^{|\alpha|}}{\alpha!} \xi^\alpha F(\eta_\alpha)$$

    where $\eta_\alpha(x) := x^\alpha \eta(x)$.

    As for growth at infinity, it must be the case that $||m_{-\xi}\eta||_{\alpha,\beta} \leq C_{\alpha,\beta}(1 + |\eta|^{|\beta|})$ (a fact easily established by induction). Since $|F(m_{-\xi}\eta)| \leq C \sum_{|\alpha|,|\beta| \leq N} ||m_{-\xi}\eta||_{\alpha,\beta}$ (by continuity of F) $\hat{F}$ must grow now faster than some positive power of $|\xi|$.

– Every compactly supported distribution is equal to a finite sum of distributional derivatives of (integrable) continuous functions with support arbitrarily near the support of the original distribution.

  Proof: Because of the polynomial growth of $\hat{F}$, the funciton $(1+4\pi^2|\xi|^2)^{-N}\hat{F}(\xi)$ is integrable for N sufficiently large; fix such an N and let $\hat{g} = (1+4\pi^2|\xi|^2)^{-N}\hat{F}(\xi)$. We know by Riemann-Lebesgue that g is a continous function, vanishing at infinity, and that $(1-\Delta)^N g = F$ in the sense of distributions (just pass the derivatives through the Fourier transform and they become multiplication).

  Moreover, g is smooth away from the support of F (this is a somewhat subtle fact that will be taken up at a later date). Thus, $(1-\Delta)^N(\eta g)$ differs from F by a smooth function of compact support (thanks to the Leibnitz rule).

- **Note.** There are functions in L^p for $p > 2$ whose Fourier transform is not a function. The following is a rough outline of the proof:

  – First prove van der Corput's Lemma: Suppose that f a real-valued function on the interval $[a, b]$ which is C^2 and has derivative f' which is nonvanishing and monotone. Then

  $$\left|\int_a^b e^{if(t)}dt\right| \leq 2\max\left\{\frac{1}{|f'(a)|}, \frac{1}{|f'(b)|}\right\}.$$

  Proof: Integrate by parts using the following trick:

  $$\int_a^b e^{if(t)}dt = \int_a^b \frac{1}{if'(t)}\frac{d}{dt}(e^{if(t)})dt = \left.\frac{e^{if(t)}}{if'(t)}\right|_a^b - \int_a^b e^{if(t)}\frac{d}{dt}\left(\frac{1}{f'(t)}\right)dt.$$

  Now put absolute values on both sides of the equality to obtain

  $$\left|\int_a^b e^{if(t)}dt\right| \leq \frac{1}{|f'(b)|} + \frac{1}{|f'(a)|} + \int_a^b \left|\frac{d}{dt}\left(\frac{1}{f'(t)}\right)\right|dt.$$

  Since f' is monotone, the integral on the right-hand side is equal to $|\frac{1}{f'(b)} - \frac{1}{f'(a)}|$.

  – Consider the functions φ_λ on $\mathbb{R}$ given by the equation

  $$\varphi_\lambda(x) := \int_0^1 e^{2\pi i(xt+\lambda t^2)}dt$$

  where λ is a positive real number. There exists a constant $C > 0$ such that, for any $\lambda > 0$ and any $p \geq 2$ (including $p = \infty$), $||\varphi_\lambda||_p \leq C\lambda^{1/p-1/2}$.

  Proof: By Plancherel, $||\varphi_\lambda||_2 = 1$ for all λ. Consider now the L^∞-norm of φ_λ. Observe that the phase $xt + \lambda t^2$ has a critical point at $t_0 = -\frac{x}{2\lambda}$. Let I_0 be the interval centered at t_0 with length $\lambda^{-\frac{1}{2}}$. Let I_l be the interval of real numbers to the left of I_0 and likewise for I_r. Now on both I_l and I_r, the phase $xt + \lambda t^2$ has nonvanishing, monotone derivative. We make the decomposition

  $$|\varphi_\lambda(x)| \leq \left|\int_{[0,1]\cap I_l} e^{2\pi i(xt+\lambda t^2)}dt\right| + \left|\int_{[0,1]\cap I_r} e^{2\pi i(xt+\lambda t^2)}dt\right| + \left|\int_{[0,1]\cap I_0} e^{2\pi i(xt+\lambda t^2)}dt\right|$$

and apply van der Corput's lemma to the first two integrals. To estimate the last integral, simply use the fact that the length of I_0 is small. The end result is that

$$|\varphi_\lambda(x)| \leq \frac{2}{4\pi\lambda^{\frac{1}{2}}} + \frac{2}{4\pi\lambda^{\frac{1}{2}}} + \lambda^{-\frac{1}{2}}.$$

This estimate is independent of x, so $||\varphi_\lambda||_\infty \leq 2\lambda^{-\frac{1}{2}}$. In general, $||\varphi_\lambda||_p \leq ||\varphi_\lambda||_2^{\frac{2}{p}} ||\varphi_\lambda||_\infty^{1-\frac{2}{p}}$ for any p in the range $[2,\infty]$.

– For each positive integer j, let $\lambda_j := (2^{\frac{3p}{p-2}})^j$. Using the estimate derived for $||\varphi_\lambda||_p$, it is straightforward to check that

$$F := \sum_{j=0}^{\infty} 2^j \varphi_{\lambda_j}$$

is a convergent sum in the topology of $L^p(\mathbb{R})$ for any p strictly larger than two. We will show that $\hat{F}$ is not equal to a function on any interval $(a,b) \subset [0,1]$.

To see this, test $\hat{F}$ against the function $\psi_\sigma(t) := 2t\psi(t^2)e^{-2\pi i\sigma t^2}$ for some smooth, nonnegative ψ compactly supported in $(0,1)$. Because the sum converges in $L^p(\mathbb{R})$, it must be the case that

$$\hat{F}(\psi_\sigma) = \sum_{j=0}^{\infty} 2^j \int e^{2\pi i(\lambda_j-\sigma)t^2} 2t\psi(t^2)dt$$

$$= \sum_{j=0}^{\infty} 2^j \int e^{2\pi i(\lambda_j-\sigma)t} \psi(t)dt = \sum_{j=0}^{\infty} 2^j \hat{\psi}(\sigma - \lambda_j)$$

(the second to last equality follows from a change of variables). Now $|\hat{\psi}(t)| \leq C_N|t|^{-N}$ for each positive integer N. Fix $\sigma = \lambda_{j_0}$ for any j_0. If $j > j_0$, then

$$|\lambda_j - \sigma| = \lambda_j(1 - 2^{\frac{3p(j_0-j)}{p-2}}) \leq \lambda_j(1 - 2^{\frac{3p}{p-2}}).$$

Likewise, if $j < j_0$, then

$$|\lambda_j - \sigma| \leq \lambda_{j_0}(1 - 2^{\frac{3p}{p-2}}).$$

Hence

$$\left|\hat{F}(\psi_{\lambda_{j_0}}) - 2^{j_0}\hat{\psi}(0)\right| \leq \sum_{j=0}^{j_0-1} 2^j C_N |\lambda_{j_0}|^{-N} + \sum_{j=j_0+1}^{\infty} 2^j C_N |\lambda_j|^{-N} \leq C_N'' 2^{j_0(1-\frac{3pN}{p-2})}$$

provided that N is chosen large enough so that the quantity $1 - \frac{3pN}{p-2}$ is negative. Letting $j_0 \to \infty$, it follows that $|\hat{F}(\psi_{\lambda_{j_0}})| \to \infty$ (which could not happen if $\hat{F}$ were an integrable function since the test functions ψ_σ are uniformly bounded).

7. Regular Homogeneous Distributions

- A regular homogeneous distribution F on the space E_n is a distribution which equals a smooth function away from the origin and satisfies the equation $\delta_a F = a^m F$ for some (possibly complex) number m and all $a > 0$ (recall that δ_a is lower dilation: $\delta_a f(x) = f(ax)$ in the case of functions).

- The class of regular homogeneous distributions is closed under the Fourier transform. In particular, if F is regular and homogeneous of degree m, then $\hat{F}$ is regular and homogeneous of degree $-n-m$.

  Proof: Homogeneity follows from the usual testing on Schwartz functions:

$$
\begin{aligned}
(\delta_a \hat{F})(\varphi) &= F((\delta^a \varphi)^\wedge) = F(\delta_a \hat{\varphi}) \\
&= a^{-n} F(\delta^{1/a} \hat{\varphi}) = a^{-n-m} (\delta_{1/a} F)(\hat{\varphi}) = a^{-n-m} F(\hat{\varphi}) = a^{-n-m} \hat{F}(\varphi)
\end{aligned}
$$

  (where we use the facts that $\delta_a = a^{-n} \delta^{1/a}$ and that F is homogeneous of degree m).

  As for regularity: Fix η be a smooth cutoff, identically one near the origin and compactly supported. Then $\hat{F} = (\eta F)^\wedge + ((1-\eta)F)^\wedge$. Let $\eta F = F_0$ and $(1-\eta)F = F_\infty$. Now F_0 is compactly supported, so its Fourier transform is a C^∞ function. The distribution F_∞, on the other hand, is a smooth function; moreover, it equals some smooth, homogeneous function outside of some compact set (the support of η) containing a neighborhood of the origin.

  Fix any multiindex α. For any $j = 1, \ldots, n$ and any large natural number N, there is an appropriate constant $c_{\alpha,j,N}$ so that $x_j^N \partial^\alpha \hat{F}_\infty = c_{\alpha,j,N} (\partial_j^N x^\alpha F_\infty)^\wedge$ in the sense of distributions. If N is chosen sufficiently large (for example, so that $-N + |\alpha| + \mathfrak{Re}(m) < -n$), then the function $\partial_j^N x^\alpha F_\infty$ is actually integrable (since it equals a smooth function homogeneous of degree $-N + |\alpha| + m$ outside some large ball). It follows that $x_j^N \partial^\alpha \hat{F}_\infty$ is a continuous function (in the sense of distributions). If N is chosen to be even, then $(\sum_{j=1}^n x_j^N) \partial^\alpha \hat{F}_\infty$ is also a function. Consequently $\partial^\alpha \hat{F}_\infty$ must also equal a continuous function away from the origin (since the sum $x_1^N + \cdots + x_n^N$ vanishes only at the origin).

  One caveat: This shows that the weak derivatives of $\hat{F}_\infty$ are always continuous functions away from the origin. To show that the function F_∞ is actually infinitely differentiable in the classical sense requires a reprise of previous arguments used to prove classical differentiability from weak differentiability (moving difference quotients, etc.) The prototype looks like this:

$$
\begin{aligned}
\int \frac{\hat{F}_\infty - \tau_{he_j} \hat{F}_\infty}{h} \varphi dx &= \int \hat{F}_\infty \frac{\varphi - \tau_{-he_j} \varphi}{h} dx = - \int \hat{F}_\infty \left(\int_0^1 \tau_{-\theta h e_j} \frac{\partial \varphi}{\partial x_j} d\theta \right) dx \\
&= - \int_0^1 \hat{F}_\infty \tau_{-\theta h e_j} \frac{\partial \varphi}{\partial x_j} dx d\theta = \int_0^1 \int \frac{\partial \hat{F}_\infty}{\partial x_j} \tau_{-\theta h e_j} \varphi dx d\theta \\
&= \int \left(\int_0^1 \tau_{\theta h e_j} \frac{\partial \hat{F}_\infty}{\partial x_j} d\theta \right) \varphi dx
\end{aligned}
$$

where φ is any Schwartz function which is compactly supported away from the origin. Since everything in sight is continuous, it follows that

$$\frac{\hat{F}_\infty - \tau_{he_j}\hat{F}_\infty}{h} = \int_0^1 \tau_{\theta h e_j} \frac{\partial \hat{F}_\infty}{\partial x_j} d\theta$$

pointwise everywere (which can be verified by taking a sequence φ_k supported on smaller and smaller neighborhoods of a single point). Now the limit as $h \to 0$ clearly exists.

- **Remark.** The only homogeneous distributions supported at the origin are linear combinations of derivatives of the Dirac delta. Consequently, all regular homogeneous distributions of degree $m \neq -n, -n-1, \ldots$ are uniquely determined by the function they equal away from the origin.

  Proof. If F is homogeneous and compactly supported, its Fourier transform $\hat{F}$ is a homogeneous, real analytic function. This is only possible if $\hat{F}$ is actually a homogeneous polynomial function. The inverse Fourier transform (in the sense of distributions) of any polynomial function, however, is a linear combination of derivatives of the Dirac delta function.

- Let O be an orthogonal matrix. If $\varphi_O(x) := \varphi(Ox)$, then $\hat{\varphi}_O = (\hat{\varphi})_O$. The rotation of a distribution is given by $F_O(\varphi) := F(\varphi_{O^{-1}})$; thus in general $(F_O)^\wedge = (\hat{F})_O$.

- **Corollary.** If $0 < \alpha < n$, the distribution $|x|^{-n+\alpha}$ has Fourier transform equal to $c|x|^{-\alpha}$ for some constant c. [This follows from rotation-invariance and the uniqueness of homogeneous distributions equal to a fixed function away from the origin.]

Aside: Some Basic Facts and Inequalities

- **Jensen's Inequality.** Let φ be convex on a closed interval (bounded or unbounded), let f be a function on E_n taking values in that interval, and let g be a nonnegative function with integral 1. Then

$$\varphi\left(\int f(x)g(x)dx\right) \leq \int \varphi(f(x))g(x)dx.$$

  Proof: If φ is convex on (a,b), then the left and right derivatives of φ exist at $c \in (a,b)$ and satisfy $\varphi_l'(c) \leq \varphi_r'(c)$. This is because, for $t < s$,

$$\frac{\varphi(x+t)-\varphi(x)}{t} \leq \frac{\varphi(x+s)-\varphi(x)}{s}$$

  (i.e., when φ is convex, the slopes of the secants from $(c, \varphi(c))$ $(c+\epsilon, \varphi(c+\epsilon))$ are decreasing as $\epsilon \to 0^+$ and bounded below by the slopes of the secants from $(c, \varphi(c))$ to $(c-\epsilon, \varphi(c-\epsilon))$ and bounded monotone sequences converge).

  Fix $c := \int f(x)g(x)dx$. If c happens to fall on the boundary of the interval on which φ is convex, then it must be the case that $f(x)g(x) = \lambda g(x)$ almost everywhere for some λ (equal to the endpoint of the interval). In this case, it's easy to see that both sides of the inequality must be equal.

  Assume, then, that c is in the interior of the interval, and choose any $m \in [\varphi_l'(c), \varphi_r'(c)]$. It follows that $m(y-c) + \varphi(c) \leq \varphi(y)$ for all y in the interval on which φ is convex. Now set $y = f(x)$, multiply by $g(x)$ and integrate with respect to x. One sees that

$$\int m(f(x)-c)g(x)dx = 0$$

  by definition of c and the fact that g has integral one. Once this term vanishes, one is left with precisely the desired inequality.

- **Minkowski's Inequality.** For $1 \leq p \leq \infty$ and $f_j \in L^p(E_n)$, $j = 1, \ldots, N$,

$$\left|\left|\sum_{j=1}^N f_j\right|\right|_p \leq \sum_{j=1}^N ||f_j||_p. \qquad (*)$$

  Proof: First assume $p < \infty$. For each j, let

$$\theta_j := \frac{||f_j||_p}{\sum_{j=1}^N ||f_j||_p} \text{ and } g_j(x) := \frac{f_j(x)}{||f_j||_p}.$$

  The relevant facts here are that the constants θ_j are nonnegative and sum to one, that each g_j has $||g_j||_p = 1$, and that the ratio of the left-hand side of $(*)$ to the right-hand side is equal to $||\sum_{j=1}^N \theta_j g_j||_p$ (and so one would like to show that this quantity is no greater than one). Now the function $\varphi(x) := |x|^p$ is convex when $p \geq 1$; therefore $|\sum_{j=1}^N \theta_j g_j(x)|^p \leq \sum_{j=1}^n \theta_j |g_j(x)|^p$. The integral over x of the right-hand side of this inequality is bounded above by one, and the integral of the left-hand side is just $||\sum_{j=1}^{N} \theta_j g_j||_p^p$; thus $(*)$ is proved.

When $p = \infty$, Minkowski's inequality follows simply from a pointwise application of the triangle inequality for complex numbers.

- **Minkowski's Integral Inequality.** Fix any function f measurable on E_{n+m} (write $f_y(x)$ where $x \in E_n$ and $y \in E_m$) and any g on E_m.

$$\left|\left|\int f_y g(y) dy\right|\right|_p = \left(\int \left|\int f_y(x) g(y) dy\right|^p dx\right)^{\frac{1}{p}} \leq \int ||f_y||_p |g(y)| dy$$

for $1 \leq p < \infty$ and analogously for $p = \infty$. In particular, when the right-hand side is finite, the integral $\int f_y(x) g(y) dy$ exists for almost every x and defines a function in $L^p(E_n)$.

Proof: First observe that for almost every $x \in E_n$, the function $f_y(x)$ of y is E_m-measurable by Fubini's theorem. Moreover, when the integral $\int f_y(x) g(y) dy$ exists for a fixed x, its magnitude is no more than $\int |f_y(x)||g(y)| dy$ which always exists as a nonnegative real number or $+\infty$ (and when the former integral fails to exist, the latter must equal $+\infty$).

For each $y \in E_m$, let

$$h(y) := \frac{||f_y||_p |g(y)|}{\int ||f_z||_p |g(z)| dz} \text{ and } F_y(x) := \frac{|f_y(x)|}{||f_y||_p}$$

(again, Fubini ensures measurability of h and F). Now h is nonnegative and has total integral one and

$$\int F_y(x) h(y) dy = \frac{\int |f_y(x)||g(y)| dy}{\int ||f_z||_p |g(z)| dz}$$

almost everywhere. Jensen's inequality gives that

$$\left|\int F_y(x) h(y) dy\right|^p \leq \int |F_y(x)|^p h(y) dy.$$

Integrating over x as in the discrete case, one obtains the inequality

$$\left(\int \left|\int |f_y(x)||g(y)| dy\right|^p dx\right)^{\frac{1}{p}} \leq \int ||f_y||_p |g(y)| dy.$$

If the right-hand side of this inequality is infinite, there is nothing to prove. Assuming it is finite, this means, in particular, that $\int |f_y(x)||g(y)| dy$ is finite for almost every x and so $\int f_y(x) g(y) dy$ must also exist for almost every x. But when it exists, it is dominated in magnitude by the integral of the absolute values, so the inequality is completed.

The case $p = \infty$ is left as an exercise.

- **Corollary.** Given a function $f \in L^p(E_n)$ and $g \in L^1(E_n)$, the convolution

$$f \star g(x) := \int f(x-y)g(y)dy$$

is defined a.e., is in $L^p(E_n)$, and satsifies $||f \star g||_p \leq ||f||_p ||g||_1$.

- **Lemma (Modulus of continuity).** Suppose $f \in L^p(E_n)$, $1 \leq p < \infty$. Then $||f - \tau_h f||_p \to 0$ as $h \to 0$.
  - Step 0: Given f, let $f = f_1 + if_2 - f_3 - if_4$ where f_1, f_2, f_3, f_4 are all nonnegative (simply restrict the real and imaginary parts of f to the sets where each is positive and negative; in this way it is also true that $||f_k||_p \leq ||f||_p$ for $k = 1, 2, 3, 4$). If $||f_k - \tau_h f_k||_p \to 0$ as $h \to 0$, then the triangle inequality guarantees that $||f - \tau_h f||_p$ as well. Therefore one may assume without loss of generality that $f \geq 0$.
  - Step 1: Given $f \in L^p$ nonnegative and ϵ, there is a continuous function $\tilde{f}$, bounded with compact support, such that $||f - \tilde{f}||_p < \epsilon/3$.
  - Step 2: By the triangle inequality:

$$||f - \tau_h f||_p \leq ||f - \tilde{f}||_p + ||\tilde{f} - \tau_h \tilde{f}||_p ||\tau_h \tilde{f} - \tau_h f||_p \leq ||\tilde{f} - \tau_h \tilde{f}||_p + 2\epsilon/3.$$

  But $\tilde{f}$ is uniformly continuous; therefore, $\tilde{f} - \tau_h \tilde{f}$ tends to zero uniformly as $h \to 0$. Moreover, $\tilde{f} - \tau_h \tilde{f}$ vanishes identically outside some fixed ball of radius R centered at the origin when $|h| \leq 1$ (that is, if R is large enough, then both terms vanish separately outside the ball). Therefore the norm $||\tilde{f} - \tau_h \tilde{f}||_p$ is less than some fixed constant (namely, the volume of the ball of radius R) times the supremum of the pointwise values of $\tilde{f} - \tau_h \tilde{f}$—this supremum must tend to zero by uniform continuity. Thus if h is sufficiently small, $||\tau_h \widetilde{f} - \widetilde{f}||_p < \epsilon/3$.

8. The Hardy-Littlewood Maximal Function

- Let $B(x,r)$ be the Euclidean ball of radius r centered at x. The Hardy-Littlewood maximal operator is the object which associates to every locally integrable function f the function Mf given by

  $$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

  The operator M is called sublinear, since $M(af+g) \leq |a| Mf + Mg$.

  Remark: If the average of $|f|$ over the ball $B(x,r)$ is strictly greater than α, then the average over $B(x',r')$ will also be strictly greater than α if x' is sufficently near x and r' is sufficently near r (for example, by dominated convergence). Therefore the function Mf is lower semicontinuous (and hence Mf is always measurable).

- The Hardy-Littlewood maximal operator is bounded on $L^\infty(E_n)$ with constant one, i.e., $||Mf||_\infty \leq ||f||_\infty$. On the other hand, if $f \in L^1(E_n)$ and $Mf \in L^1(E_n)$, then it must be the case that $f \equiv 0$. If f is not identically zero, one can show that there is a constant C depending on f such that $Mf(x) \geq C|x|^{-n}$ for x sufficiently large, so Mf is not integrable.

- The goal of this section is to prove that, for any $p > 1$, there is a constant C_p such that $||Mf||_p \leq C||f||_p$ for all $f \in L^p(E_n)$. When $p = 1$, there is a replacement inequality (called a weak-(1,1) inequality): for any $\lambda > 0$,

  $$|\{x \in E_n \ | \ Mf(x) > \lambda\}| \leq \frac{C||f||_1}{\lambda}$$

  (observe that if it were the case that $||Mf||_1 \leq C||f||_1$, then the weak-(1,1) inequality would follow trivially).

- **Distribution functions.** Given any locally integrable function f, let $\Phi_f(\lambda) := |\{x \in E_n \ | \ |f(x)| > \lambda\}|$. The following equality will come in useful along the way:

  $$\int |f(x)|^p dx = p \int_0^\infty \lambda^{p-1} \Phi_f(\lambda) d\lambda.$$

  To see that this is true, suppose first that f is a simple function (i.e., a finite linear combination of characteristic functions of disjoint sets of finite measure), then take a sequence of simple functions increasing monotonically to $|f|$. Using the standard measure-theoretic arguments, the distribution functions of the simple functions also increase to Φ_f. The details are left as an exercise.

- **Vitali Covering Lemma.** Let $K \subset E_n$ be compact and covered by a family of balls $B(x_j, r_j)$ for $j = 1, \ldots, N$. Then there exists a subcollection of these balls which are pairwise disjoint and whose triples $(B(x,r))^{***} := B(x, 3r)$ cover K.

  Proof: Let $B(x_1, r_1)$ be any ball in the collection with a maximal radius. Next take $B(x_2, r_2)$ to be any ball among those not intersecting $B(x_1, r_1)$ which has a maximal radius. Inductively, take $B(x_k, r_k)$ to be any ball among those not intersecting

$\bigcup_{j=1}^{k-1} B(x_j, r_j)$ which has a maximal radius. Eventually this process must terminate; suppose that the final ball is chosen at step $k = M$ for some M. These balls $B(x_k, r_k)$ for $k = 1, \ldots, M$ satisfy the conditions of the lemma.

Suppose there is some ball $B(x, r)$ which is part of the original collection but not the subcollection. Since it is not part of the subcollection, there must be some index k so that $B(x, r) \cap B(x_k, r_k) \neq \emptyset$; take k minimal. Since the ball $B(x, r)$ was not added to the subcollection at step $k - 1$, it must be the case that $r \leq r_k$. Since $B(x, r) \cap B(x_k, r_k) \neq \emptyset$ and $r \leq r_k$, it must be the case that $B(x, r) \subset B(x_k, 3r_k)$ by the triangle inequality. Therefore the union of the triples of all balls in the subcollection must, in fact, cover K.

- **Proof of the Weak-(1,1) Inequality.** Suppose that $f \in L^1(E_n)$. Fix some $\lambda > 0$ and consider the set $E_\lambda := \{x \in E_n \mid Mf(x) > \lambda\}$. Let $K \subset E_\lambda$ be any compact set. By definition of the maximal operator, the set K may be covered by balls $B(x, r)$ on which
$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x)| dx > \lambda,$$
and since K is compact, there is a finite subcover. By the Vitali covering lemma, there exist balls B_k, $k = 1, \ldots, M$ in this cover which are pairwise disjoint and whose triples cover K. Combining all these facts,
$$|K| \leq \sum_{j=1}^{M} |B_j^{***}| = \sum_{j=1}^{M} 3^n |B_j| \leq \sum_{j=1}^{M} \frac{3^n}{\lambda} \int_{B_j} |f(x)| dx \leq \frac{3^n}{\lambda} ||f||_1$$
(the first inequality follows from the fact that the triples cover; the second inequality follows from the fact that the averages of $|f|$ are greater than λ, and the third from the fact that the balls are disjoint). Since E_λ may be approximated from below by compact sets, it must be the case that
$$\Phi_f(\lambda) \leq \frac{3^n}{\lambda} ||f||_1.$$

- **Interpolation.** Suppose that $f = f_1 + f_2$. Observe that the set where $Mf(x) > \lambda$ must be contained in the union of the sets where each of $Mf_1(x)$ and $Mf_2(x)$ are bigger than $\lambda/2$. Taking $f_1 = f$ when $|f| > \lambda/2$ and zero otherwise, the set where $M(f_2) > \lambda/2$ is actually empty (since $|f_2| \leq \lambda/2$ and $||Mf_2||_\infty \leq ||f_2||_\infty$). Therefore, the inequality
$$\Phi_{Mf}(\lambda) \leq \Phi_{Mf_1}\left(\frac{\lambda}{2}\right)$$
holds (where f_1 depends on λ). Therefore
$$p \int_0^\infty \lambda^{p-1} \Phi_{Mf}(\lambda) d\lambda \leq p \int_0^\infty \Phi_{Mf_1}\left(\frac{\lambda}{2}\right) d\lambda.$$
By the weak-(1,1) inequality $|\Phi_{Mf_1}(\lambda/2)| \leq 2 \cdot 3^n \lambda^{-1} ||f_1||_1$. Therefore
$$p \int_0^\infty \lambda^{p-1} \Phi_f(\lambda) d\lambda \leq 2 \cdot 3^n p \int_0^\infty \lambda^{p-2} \int \chi_{|f(x)| > \frac{\lambda}{2}}(x) |f(x)| dx d\lambda.$$

Now change the order of integration; for each fixed x, the integral over λ may be computed explicitly (since the range of integration is $\lambda \in [0, 2|f(x)|]$. The result is that

$$||Mf||_p^p \leq 2^p 3^n \frac{p}{p-1} \int |f(x)|^p dx.$$

- **Lebesgue Points.** A Lebesgue point of a function $f \in L^p(E_n)$ is a point x_0 for which

$$\lim_{r \to 0} \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} |f(y) - f(x_0)| dy \to 0 \qquad (*)$$

(this is a slight strengthening of the statement that the averages over balls tend to the value at the center). We will now prove that almost every point of f is a Lebesgue point for any $1 \leq p \leq \infty$. First observe that the limit exists and equals zero at every point if f happens to be a continuous function. By the triangle inequality, then, for any point x and any continuous function g,

$$\begin{aligned} \limsup_{r \to 0} \frac{1}{|B(x,r)|} & \int_{B(x,r)} |f(y) - f(x)| dy \qquad (**) \\ & \leq \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |(f(y) - g(y)) - (f(x) - g(x))| dy \\ & \leq M(f-g)(x) + |f(x) - g(x)|. \end{aligned}$$

Fix any sequence g_j of continuous functions tending to f in the L^p-norm for $1 \leq p < \infty$. Let $E_{\epsilon,j}$ be the set on which $M(f - g_j)(x) + |f(x) - g_j(x)|$ is greater than ϵ. By the weak-(1,1) inequality (or the L^p-boundedness plus Tchebyshev), $|E_{\epsilon,j}| \leq C\epsilon^{-p}||f - g_j||_p^p$, which tends to zero as $j \to \infty$. Since the limsup of the averages (that is, $(**)$) can only be greater than ϵ at points in $\bigcap_j E_{\epsilon,j}$, it follows that this can only occur on a set of measure zero. Fixing $\epsilon \to 0^+$ through some countable sequence, it follows that the limsup must be zero almost everywhere.

- Given a function $f \in L^p(E_n)$ for $1 \leq p \leq 2$, $\hat{f}$ exists as a function (which is the sum of $\hat{f}_1 \in C_0(E_n)$ and $\hat{f}_2 \in L^2(E_n)$). The question to be considered next is in what sense the Fourier inversion formula holds for $\hat{f}$. Since the usual integral is not defined, one must make sense of it as a limit of integrals which are well-defined. There is an infinite variety of ways to do this (all going under the heading of summability methods). Two of the most important classical examples are

$$G_\epsilon f(x) := \int e^{2\pi i x\cdot\xi} \hat{f}(\xi) e^{-\pi\epsilon|\xi|^2} d\xi,$$
$$P_\epsilon f(x) := \int e^{2\pi i x\cdot\xi} \hat{f}(\xi) e^{-2\pi\epsilon|\xi|} d\xi.$$

The former are called Gauss-Weierstrass means, and the latter are known as Abel means. The hope is that, as $\epsilon \to 0^+$, $G_\epsilon f \to f$ and $P_\epsilon f \to f$ in some sense (or maybe even in more than one sense). By the multiplication formula for the Fourier transform, one expects that both functions $G_\epsilon f$ and $P_\epsilon f$ can be written as convolutions of f with some functions whose Fourier transforms are $e^{-\pi\epsilon|\xi|^2}$ and $e^{-2\pi\epsilon|\xi|}$, respectively.

- **Lemma.** For any positive constant α and any dimension n,

$$\int_{E_n} e^{-2\pi i x\cdot\xi} e^{-\pi\alpha|x|^2} dx = \alpha^{-n/2} e^{-\pi|\xi|^2/\alpha},$$
$$\int_{E_n} e^{-2\pi i x\cdot\xi} e^{-2\pi\alpha|x|} dx = \frac{c_n \alpha}{(\alpha^2 + |\xi|^2)^{(n+1)/2}},$$

where $c_n := \Gamma(\frac{n+1}{2})\pi^{-(n+1)/2}$.

Proof. The first inequality has already been established. To establish the second, there is an intermediate identity which must first be shown (and is interesting in its own right):

$$e^{-2\pi\beta} = \int_0^\infty \frac{e^{-\pi u}}{\sqrt{u}} e^{-\pi\beta^2/u} du, \qquad \beta > 0. \tag{$*$}$$

This identity is an example of what's known as the subordination principle (many functions are "subordinate" to Gaussians in the sense that they may be expressed in terms of Gaussians). To prove $(*)$, one begins with two simplier identities

$$e^{-2\pi\beta} = \frac{1}{\pi} \int_{-\infty}^\infty \frac{e^{-2\pi i\beta z}}{1+z^2} dz,$$
$$\frac{1}{1+z^2} = \pi \int_0^\infty e^{-\pi(1+z^2)u} du$$

(here assuming $\beta > 0$). The second identity is immediate, and the first is only slightly more complicated—when interpreted as a contour integral, closing the contour over a large circular arc in the lower half-plane and using the residue theorem gives the answer. Substituting the latter identity into the former and changing the order of integration gives that

$$e^{-2\pi\beta} = \int_0^\infty \int_{-\infty}^\infty e^{-2\pi i\beta z - \pi(1+z^2)u} dz du;$$

now the inner integral may be evaluated (it's just the Fourier transform of a 1-D Gaussian), and the result is precisely $(*)$. To return to the original computation, the function $e^{-2\pi\alpha|x|}$ may be written as an integral of Gaussians by the subordination principle. Taking Fourier transforms gives

$$\begin{aligned}
\int e^{-2\pi i x\cdot\xi} e^{-2\pi\alpha|x|} dx &= \int_0^\infty \frac{e^{-\pi u}}{\sqrt{u}} \int e^{-2\pi i x\cdot\xi} e^{-\pi\alpha^2|x|^2/u} dx du \\
&= \alpha^{-n} \int_0^\infty u^{(n-1)/2} e^{-\pi u(1+|\xi|^2/\alpha^2)} du \\
&= \frac{\alpha\Gamma(\frac{n+1}{2})\pi^{-(n+1)/2}}{(\alpha^2+|\xi|^2)^{(n+1)/2}}.
\end{aligned}$$

- Now compare Poisson and Gaussian convolutions to the H-L Maximal Function: If Φ is a radial, decreasing piecewise-C^1 function and f is a simple function, then

$$\begin{aligned}
\int |f(x)|\Phi(|x|)dx &= -\int_0^\infty \Phi'(R) \int_{B(0,R)} |f(x)| dx dR \\
&\leq -\int_0^\infty \Phi'(R) Mf(0)|B(0,R)| dR = ||\Phi||_1 Mf(0).
\end{aligned}$$

  Consequently, it follows that $|f \star \varphi(x)| \leq Mf(x)$ whenever φ is radial, decreasing, and has total integral 1. In particular, then

$$\begin{aligned}
&\sup_{\epsilon>0} |G_\epsilon f(x)| \leq Mf(x), \\
&\sup_{\epsilon>0} |P_\epsilon f(x)| \leq Mf(x).
\end{aligned}$$

- Reprising the argument for the existence of Lebesgue points, it follows that for any $f \in L^p(E_n)$, $1 \leq p < \infty$, one has $G_\epsilon f(x) \to f(x)$ for almost every x as $\epsilon \to 0$ (and likewise for $P_\epsilon f(x)$).

- It is also true that $G_\epsilon f$ and $P_\epsilon f$ tend to f in the L^p-sense as $\epsilon \to 0$. The proof is left as an exercise (Consider the difference $G_\epsilon f - f$ as an L^p-function and use the fact that $||\tau_h f - f||_p \to 0$ as $h \to 0$ plus the approximate identity properties of the kernels of G_ϵ).

- Let $u(x,t) := P_t f(x)$ and $v(x,t) := G_t f(x)$ for $x \in E_n$ and $t > 0$. An easy exercise differentiating integrals gives that

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}(x,t) = 0$$

$$\frac{\partial v}{\partial t}(x,t) = \frac{1}{4\pi} \sum_{j=1}^n \frac{\partial^2 v}{\partial x_j^2}(x,t)$$

  The former PDE is called the Laplace equation (on the upper half-space of E_{n+1}) and the latter is called the heat equation. We have shown that these equations have solutions with boundary value equal to f almost everywhere.

The Riesz-Thorin Interpolation Theorem

- The premise of the Reisz-Thorin interpolation theorem is as follows: you have a single linear operator T which is bounded in several different ways (like the Fourier transform, for which one has the Riemann-Lebesgue lemma as well as Plancherel's theorem). For completely abstract reasons, the inequalities already proved can be used to deduce even more inequalities that the operator must follow.

- Suppose that you're given bounded linear operators

$$T_j : L^{p_j}(E_n) \to L^{q_j}(E_n) \text{ for } j = 0, 1$$

which agree on the overlap, that is, $T_0 f = T_1 f$ for all $f \in L^{p_0}(E_n) \cap L^{p_1}(E_n)$ (the operators T_0 and T_1 are thus thought of as being different instances of the same operator, like the Fourier transform, which has *a priori* different defintions on $L^1(E_n)$ and $L^2(E_n)$).

- We've already seen that T_0 and T_1 can be uniquely extended to an operator T mapping $L^p(E_n)$ to $L^{q_1}(E_n) + L^{q_2}(E_n)$ when p is between p_1 and p_2 (by taking $Tf = Tf_0 + Tf_1$ where $f_j \in L^{p_j}(E_n)$ and $f_0 + f_1 = f$; the agreement on the overlap shows that the sum $Tf_0 + Tf_1$ is independent of the particular spltting of f). In fact, a stronger statement is true of this operator T:

- **Theorem (Riesz-Thorin).** $T : L^p(E_n) \to L^q(E_n)$ if there is some $\theta \in [0, 1]$ such that $\frac{\theta}{p_0} + \frac{1-\theta}{p_1} = \frac{1}{p}$ and $\frac{\theta}{q_0} + \frac{1-\theta}{q_1} = \frac{1}{q}$. Moreover, if

$$||T_j f||_{q_j} \leq C_j ||f||_{p_j} \text{ for } j = 0, 1 \text{ and all } f \in L^{p_j}(E_n)$$

then

$$||Tf||_q \leq C_0^{\theta} C_1^{1-\theta} ||f||_p \text{ for all } f \in L^p(E_n).$$

- **Corollary.** Given any linear operator T, let $\mathcal{R}(T) \subset [0, 1]^2$ be the set of ordered pairs $(\frac{1}{p}, \frac{1}{q})$ such that T maps $L^p(E_n)$ to $L^q(E_n)$. By the Riesz-Thorin interpolation theorem, $\mathcal{R}(T)$ must be convex. (The set $\mathcal{R}(T)$ is sometimes called the type set or Riesz diagram of T.)

- **Phragmen-Lindelöf Lemma.** Suppose F is a bounded analytic function on the strip $S := \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$. If $|F(0 + iy)| \leq k_0$ and $|F(1 + iy)| \leq k_1$, then $|F(x + iy)| \leq k_0^{1-x} k_1^x$.

  [Remark: The Phragmen-Lindelöf lemma's strength lies in the fact that one only knows *a priori* that the function F is bounded by *something*, and the conclusion is that a specific quantity for the upper bound is obtained. In fact, one does not even need to assume that F is bounded, only that

$$e^{-a|y|} \log |F(x + iy)|$$

is bounded above for some $a < \pi$. The example $F(z) = e^{e^{i\pi(z+\frac{1}{2})}}$ shows that this restriction is sharp.]

Proof of the lemma: consider the function $G_n(z) := (k_0^{z-1}k_1^{-z})e^{(z^2-1)/n}F(z)$. For each n, $G_n(z)$ is analytic on the strip S; furthermore, $|G_n(z)| \leq 1$ whenever $\mathrm{Re}(z) = 0, 1$. Since F is bounded and $e^{(z^2-1)/n}0$ uniformly as $|y| \to \infty$, it must be the case that the maximum of $|G_n(z)|$ is obtained by some $z \in S$. Since G_n is analytic, the maximum modulus principle says that the maximum of $|G_n(z)|$ on $S_M := S \cap \{x + iy \mid |y| \leq M\}$ must be attained on the boundary. If M is sufficiently large, the maximum of $|G_n(z)|$ on S_M must equal the maximum of $|G_n(z)|$ on S, and it must be attained on the boundary of S itself. On the boundary of S, $|G_n(z)| \leq 1$, so $|G_n(z)| \leq 1$ on all of S. Taking $n \to \infty$ gives the desired conclusion for F.

- Proof of Riesz-Thorin. (We leave the cases $p = \infty$ and $q = 1$ as exercises). Let f and g be any simple functions; without loss of generality, there exist positive real numbers a_j, φ_j and disjoint sets of finite measure F_j so that $f = \sum_j a_j e^{i\varphi_j}\chi_{F_j}$; likewise $g = \sum_k b_k e^{i\psi_k}\chi_{G_k}$. Let $\alpha(z) := \frac{p}{p_0}z + \frac{p}{p_1}(1-z)$ and $\beta(z) := \frac{q'}{q'_0}z + \frac{q'}{q'_1}(1-z)$ (where $\frac{1}{q} + \frac{1}{q'} = 1$ and likewise for q'_0 and so on).

  Consider the analytic function $I(z)$ given by

$$I(z) := \sum_{j,k} a_j^{\alpha(z)} b_k^{\beta(z)} e^{i(\varphi_j + \psi_k)} \int T\chi_{F_j}(x)\chi_{G_k}(x)dx.$$

  First of all, $\int T\chi_{F_j}(x)\chi_{G_k}(x)dx$ exists for any pair j, k (in fact, Hölder's inequality and the boundedness of T_0 and T_1 imply that

$$\left| \int T\chi_{F_j}(x)\chi_{G_k}(x)dx \right| \leq C_l |F_j|^{\frac{1}{p_l}} |G_k|^{\frac{1}{q'_l}} \text{ for } l = 0, 1.$$

  Therefore $I(z)$ must be bounded on the strip $S := \{x + iy \mid 0 \leq x \leq 1\}$, but the bound depends *a priori* on the functions f and g. The goal is to use the Phragmen-Lindelöf lemma to remove this dependence. The reason that one should care about $I(z)$ on the interior of the strip is that $I(\theta + 0i) = \int Tf(x)g(x)dx$ where θ is the number in $[0, 1]$ coming from the hypotheses of the theorem.

  Observe that $I(0 + iy) = \int Tf_y(x)g_y(x)dx$, where

$$f_y := \sum_j a_j^{\frac{p}{p_1}} \left(a_j^{iy\frac{p}{p_1}} e^{i\varphi_j} \right) \chi_{F_j},$$

$$g_y := \sum_k b_k^{\frac{q'}{q'_1}} \left(b_j^{iy\frac{q'}{q'_1}} e^{i\varphi_k} \right) \chi_{G_k}.$$

  Since T maps L^{p_1} to L^{q_1} with constant C_1, Hölder's inequality implies that $|I(0+iy)| \leq C_1 ||f_y||_{p_1} ||g_y||_{q'_1}$. Observe, though, that $||f_y||_{p_1} = ||f||_p^{p/p_1}$ and $||g_y||_{q'_1} = ||g||_{q'}^{q'/q'_1}$. It thus follows that one has the inequality

$$|I(0 + iy)| \leq C_1 ||f||_p^{\frac{p}{p_1}} ||g||_{q'}^{\frac{q'}{q'_1}}.$$

Likewise one can show that

$$|I(1+iy)| \leq C_0 ||f||_p^{\frac{p}{p_0}} ||g||_{q'}^{\frac{q'}{q'_0}}.$$

Therefore the Phragmen-Lindelöf lemma implies that

$$\left| \int Tf(x)g(x)dx \right| \leq C_0^\theta C_1^{1-\theta} ||f||_p ||g||_{q'}$$

for any two simple functions f and g. This is the key result of the proof—the rest of the reasoning deals with the technicalities involved with removing the assumption that f and g are simple.

- Step 1: Show that Tf must actually be a function in $L^q(E_n)$. If this were not the case, then either $\mathrm{Re}\,(Tf)(x)$ or $\mathrm{Im}\,(Tf)(x)$ is not in $L^q(E_n)$. Without loss of generality, assume that the former occurs (or replace f by if). Let g be any nonnegative simple function satisfying the pointwise inequality $|g(x)| \leq |\mathrm{Re}\,(Tf)(x)|^{q-1}$. Then one has

  $$|g(x)|^{\frac{q}{q-1}} \leq \mathrm{Re}\,(Tf)(x)\, sgn(\mathrm{Re}\,(Tf)(x))g(x)$$

  as well. The advantage is that $h(x) := sgn(\mathrm{Re}\,(Tf)(x))g(x)$ is a simple function with the same L^p-norms as g. Integrating over x, then,

  $$\int |g(x)|^{\frac{q}{q-1}} dx \leq \left| \int Tf(x)h(x)dx \right| \leq C_0^\theta C_1^{1-\theta} ||f||_p ||g||_{q'}$$

  But $q' = \frac{q}{q-1}$. *Since the norms of g are finite*, we may divide both sides by $||g||_{q'}$ to obtain the inequality that $||g||_{q'}^{q'/q} \leq C_0^\theta C_1^{1-\theta} ||f||_p$. Now replace g by a seqence of simple functions converging monotonically to $|\mathrm{Re}\,(Tf)(x)|^{q-1}|$. The monotone convergence theorem gives that $||\mathrm{Re}\,(Tf)||_q \leq C_0^\theta C_1^{1-\theta} ||f||_p$. Therefore $||Tf||_q$ must be finite. (Note that the case $q = \infty$ is similar and left as an exercise.)
- Step 2: Show that $||Tf||_q \leq C_0^\theta C_1^{1-\theta} ||f||_p$ (that is, T maps simple functions f boundedly into L^q). Since $||Tf||_q$ is finite, $h(x) := |Tf(x)|^{q-1} \frac{Tf(x)}{|Tf(x)|}$ has a finite norm in $L^{q'}$ (define this function to be zero whenever $Tf(x) = 0$). Take g_k to be a seqence of simple functions tending to h in the $L^{q'}$-norm. Then

  $$||Tf||_q^q = \int Tf(x)h(x)dx = \lim_{k\to\infty} \left| \int Tf(x)g_k(x)dx \right|$$

  (because $||Tf||_q$ is finite, Hölder's inequality allows the integral and the limit to be interchanged). But

  $$\left| \int Tf(x)g_k(x)dx \right| \leq C_0^\theta C_1^{1-\theta} ||f||_p ||g_k||_{q'}$$

  for each k, and $||g_k||_{q'} \to ||Tf||_q^{q/q'}$ (which is finite) as $k \to \infty$, therefore

  $$||Tf||_q^q \leq C_0^\theta C_1^{1-\theta} ||f||_p ||Tf||_q^{q/q'}.$$

  Since $||Tf||_q$ is finite, you can divide by $||Tf||_q^{q/q'}$ to get $||Tf||_q \leq C_0^\theta C_1^{1-\theta} ||f||_p$. (Again, $q = \infty$ is left as an exercise.)

– Step 3: Remove the restriction that f is a simple function. Given any function $f \in L^p(E_n)$, there exist nonnegative functions $f_0, \ldots, f_7$ such that $f = \sum_{j=0}^{7} i^j f_j$ and $|f_j| \leq 1$ for $j = 0, \ldots, 3$ and $|f_j(x)|^2 \geq |f_j(x)|$ for $j = 4, \ldots 7$ (that is, f_j has no values in the interval $(0, 1)$). Note that these functions f_j result from splitting f into real and imaginary parts, then splitting into positive and negative, then splitting into magnitude greater than and less than one. Consequently $||f_j||_p \leq ||f||_p$ and $||f_j||_{p_1} < \infty$ for $j = 0, \ldots, 3$ and $||f_j||_{p_0} < \infty$ for $j = 4, \ldots, 7$ (here we assume $p_0 \leq p \leq p_1$). Now let g_j^k be a monotone increasing sequence of simple functions tending to f_j as $k \to \infty$ for each j. By dominated convergence, $g_j^k \to f_j$ in L^p as well, and in either p_0 or p_1 depending on j. Thus

$$\sum_{j=0}^{3} i^j g_j^k \to \sum_{j=0}^{3} i^j f_j \text{ in } L^p(E_n) \text{ and } L^{p_1}(E_n)$$

$$\sum_{j=4}^{7} i^j g_j^k \to \sum_{j=4}^{7} i^j f_j \text{ in } L^p(E_n) \text{ and } L^{p_0}(E_n)$$

Therefore, since T_1 and T_0 are bounded,

$$T_1\left(\sum_{j=0}^{3} i^j g_j^k\right) \to T_1\left(\sum_{j=0}^{3} i^j f_j\right) \text{ in } L^{q_1}(E_n)$$

$$T_0\left(\sum_{j=4}^{7} i^j g_j^k\right) \to T_0\left(\sum_{j=4}^{7} i^j f_j\right) \text{ in } L^{q_0}(E_n)$$

by taking an appropriate subsequence, we may assume that the convergence is pointwise almost everywhere as well (this is a general property of any convergent sequence of L^q-functions).

As we've defined T,

$$Tf = T_1\left(\sum_{j=0}^{3} i^j f_j\right) + T_0\left(\sum_{j=4}^{7} i^j f_j\right) = \lim_{k\to\infty} T\left(\sum_{j=0}^{7} i^j g_j^k\right)$$

(where the limit is pointwise over some subsequence; note that $T_0 g_j^k = T_1 g_j^k = T g_j^k$ since these are simple functions). Now Fatou's theorem gives that

$$||Tf||_q \leq \liminf_{k\to\infty} \left|\left|T\left(\sum_{j=0}^{7} i^j g_j^k\right)\right|\right|_q$$

(for k in some subsequence). *Now* we know the right-hand side by step 2, since it contains only simple functions. The consequence is that

$$\left|\left|T\left(\sum_{j=0}^{7} i^j g_j^k\right)\right|\right|_q \leq C_0^\theta C_1^{1-\theta} \left|\left|\sum_{j=0}^{7} i^j g_j^k\right|\right|_p .$$

As $k \to \infty$, the right-hand side tends to $C_0^\theta C_1^{1-\theta} ||f||_p$ because the functions converge in L^p. Finally, the theorem is complete.

- **Corollary (Hausdorff-Young Inequality).** The Riemann-Lebesgue lemma and the Plancherel identity state that the following inequalities hold for the Fourier transform:

$$||\hat{f}||_\infty \leq ||f||_1$$

$$||\hat{f}||_w \leq ||f||_2$$

By Riesz-Thorin, for any $1 \leq p \leq 2$,

$$||\hat{f}||_{p'} \leq ||f||_p,$$

where $\frac{1}{p'} + \frac{1}{p} = 1$.

- **Corollary (Young's Inequality).** Given $f \in L^p(E_n)$, $g \in L^q(E_n)$, then $f \star g$ is well-defined as a function in $L^r(E_n)$ and

$$||f \star g||_r \leq ||f||_p ||g||_q$$

provided $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Proof: Fix f and let $Tg := f \star g$. The Minkowski integral inequality gives that $||Tg||_p \leq ||f||_p ||g||_1$ (that is, T maps L^1 to L^p with norm $||f||_p$). Likewise, by Hölder, $|Tg(x)| \leq ||f||_p ||g||_{p'}$ (T maps $L^{p'}$ to L^∞ with norm again $||f||_p$). Interpolation gives the stated inequalities.