Regularity of the Fourier transform on spaces of homogeneous distributions

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November 8, 2005

1. Introduction. In this paper we consider the action of the Fourier transform on spaces of regular homogeneous distributions on $\mathbb{R}^n$, by which we mean to include the nonisotropic (sometimes called quasihomogeneous [6]) case. Specifically, consider the group of dilations of $\mathbb{R}^n$ with positive scaling exponents $d_1, d_2, \ldots, d_n$ sending $(x_1, x_2, \ldots, x_n) \mapsto (\delta^{d_1} x_1, \delta^{d_2} x_2, \ldots, \delta^{d_n} x_n)$ for $\delta > 0$ (we will write this map as $\delta \circ x$ for short). Dilation induces scaling operators on functions: $\varphi \mapsto \varphi_\delta$, given by $\varphi_\delta(x) := \varphi(\delta \circ x)$. The dual scalings $\varphi \mapsto \varphi^\delta$ are given by $\varphi^\delta(x) := \delta^{-Q} \varphi(\delta^{-1} \circ x)$, where $Q := d_1 + d_2 + \cdots + d_n$. A tempered distribution F is said to be homogeneous of degree z if $F(\varphi^\delta) = \delta^z F(\varphi)$ for all Schwartz functions φ and all $\delta > 0$. In addition, a homogeneous distribution F is called regular when it is equal to a C^∞ function away from the origin.

It is well-known that the class of regular homogeneous distributions is preserved by the Fourier transform (for this and other classical results, see Fabes and Riviére [1], Gårding [2], Krée [3]). Up to a multiplicative constant, the only non-trivial regular homogeneous distributions which vanish outside the origin are the Dirac delta function D and all mixed derivatives of D. Thus any regular homogeneous F is uniquely determined by its values on the unit sphere provided the degree of homogeneity is not in a countable, discrete subset of $\mathbb{R}$ (which is, for example, contained in the set of real numbers less than or equal to $-Q$). The purpose of this paper is to study the connection between the regularity of F on the unit sphere and that of $\widehat{F}$. Our first result of this type is closely related to the Hausdorff-Young inequality (the Lebesgue spaces $L^p(S^{n-1})$ are with respect to the measure $d\sigma$ induced by the polar coordinate formula in the next section). In all the theorems that follow, τ is an arbitrary real parameter.

Theorem 1 (a la Hausdorff-Young)**.** *Fix any $1 \leq p \leq 2$, and let p' be the dual exponent. When $z = -\frac{Q}{p} + i\tau$ (and $z \neq -Q$), there exists a finite constant $C(z)$ such that any regular homogeneous distribution F of degree z satisfies*

$$||\widehat{F}||_{L^{p'}(S^{n-1})} \leq C(z) ||F||_{L^p(S^{n-1})}. \quad (1)$$

Theorem 1 is, in fact, a consequence of the Hausdorff-Young inequality when $1 < p < 2$. By generalized Marcinkiewicz interpolation, the Fourier transform maps weak-L^p into weak-$L^{p'}$ for $1 < p < 2$, and a regular homogeneous distribution F of degree $-\frac{Q}{p} + i\tau$ is in weak-L^p precisely when $||F||_{L^p(S^{n-1})} < \infty$. Thus, we are primarily concerned with the endpoint cases:

Theorem 2 (a la Plancherel)**.** *If F is a regular homogeneous distribution of degree $z = -\frac{Q}{2} + i\tau$, then*

$$||\widehat{F}||_{L^2(S^{n-1})} = ||F||_{L^2(S^{n-1})} \tag{2}$$

Theorem 3. *There exists $C > 0$ such that, for any regular homogeneous distribution F of degree z with $z = -Q + i\tau$,*

$$||\widehat{F}||_{L^\infty(S^{n-1})} \leq C(|\tau|^{-1} + |\tau|^{-\frac{1}{m+1}})||F||_{L^1(S^{n-1})} \tag{3}$$

where m is the number of distinct exponents d_j.

It is worth mentioning that, by interpolating theorems 2 and 3, one could prove a slightly stronger version of theorem 1, in which the constant appearing in (1) decays like $|\tau|$ to the power $\frac{2}{m+1}(\frac{1}{p} - \frac{1}{2})$ as $|\tau| \to \infty$.

The second result of this paper concerns L^p-Sobolev smoothing estimates for the Fourier transform, of which there are no classical analogues for functions. Results of this type have long been known for isotropically homogeneous distributions, culminating with the work of C. Lemoine [4]. Such results, however, are somewhat imcomplete because of the absence of any previous observation like theorem 1. Furthermore, these results rely heavily on a spherical harmonic decomposition of homogeneous distributions, a technique which seems to be fruitless in the nonisotropic case (since the class of such distributions is not closed under rotations). We will define suitable nonisotropic Sobolev spaces L^p_s for $s \geq 0$ on the unit sphere, which coincide with the usual Sobolev spaces in the case of isotropic dilations, and show that the following inequality holds:

Theorem 4. *Suppose $z \in \mathbb{C}$ is not equal to the degree of homogeneity for any mixed derivative of a delta function $\partial^\alpha D$ (so that $\widehat{F}$ on the unit sphere is uniquely determined by F on the unit sphere). Let p be any exponent satisfying $1 < p \leq 2$, let $s = -\operatorname{Re}(z) - \frac{Q}{p}$, and let $t \geq \max\{0, -s\}$. Then there exists a constant C such that, for any regular homogeneous distribution F of degree z,*

$$||\widehat{F}||_{L^{p'}_{s+t}(S^{n-1})} \leq C||F||_{L^p_t(S^{n-1})} \tag{4}$$

If z is not real, the same is true for $p = 1$.

2. Preliminaries. There is a unique norm function $\rho(x)$ which is C^∞ on $\mathbb{R}^n \backslash \{0\}$, identically one on the unit sphere, and satisfies $\rho(\delta \circ x) = \delta\rho(x)$ for all positive δ. Furthermore, every $x \in \mathbb{R}^n \backslash \{0\}$ may be written uniquely as $\rho(x) \circ \gamma$ for some $\gamma \in S^{n-1}$, which leads to the polar coordinate formula $\int f(x)dx = \iint_0^\infty f(\rho \circ \gamma)\rho^{Q-1} d\rho d\sigma(\gamma)$ for some appropriate measure $d\sigma$ of smooth density on the unit sphere. Proofs of these statements may be found in Fabes and Riviére [1]. Observe that the measure $d\sigma$ is invariant under the change $\gamma \mapsto -\gamma$.

Let f be a smooth function on S^{n-1}. For $\mathrm{Re}(z) > -Q$, one may extend f to a regular homogeneous distribution of degree z by the polar coordinate formula:

$$E_z f(\varphi) := \iint_0^\infty f(\gamma)\rho^z \varphi(\rho \circ \gamma)\ \rho^{Q-1} d\rho d\sigma(\gamma).$$

The family of distributions $E_z f$ is (weakly) analytic; furthermore, it may be continued meromorphically to the entire complex plane via the continuation formula (derived by an integration by parts in the ρ variable)

$$E_z f(\varphi) = -\frac{1}{z+Q} \sum_{j=1}^n d_j E_{z+d_j}(\gamma_j f)(\partial_{x_j}\varphi). \tag{5}$$

One sees immediately that the poles of $E_z f$ must are of the form $-Q - \alpha \cdot d$ for some multi-index α (precisely the degree of homogeneity of $\partial^\alpha D$ for the Dirac delta function D); let $\mathcal{C}$ be the set of all $z \in \mathbb{C}$ not of this form. For any regular homogeneous distribution F of degree $z \in \mathcal{C}$ which is equal to f on S^{n-1}, $F = E_z f$. As already mentioned, $(E_z f)^\wedge$ is a regular homogeneous distribution of degree $-Q-z$. For $z \in \mathcal{C}$, then, let $\mathcal{F}_z$ be the operator taking the smooth function f on S^{n-1} to the values of $(E_z f)^\wedge$ on S^{n-1}. Just as $F = E_z f$, $\widehat{F} = E_{-Q-z}\mathcal{F}_z f$.

From here on, we fix a smooth, monotone decreasing cutoff function η on the non-negative half line for which $\eta(t) = 1$ for all $t \in [0,1]$ and $\eta(t) = 0$ for all $t \geq 2$. The cutoff function η induces a radial cutoff function in $\mathbb{R}^n$: $\eta(\rho(x))$. We will abuse notation and sometimes write η in place of the composition $\eta(\rho(x))$.

Lemma 1. *Given any regular distribution F which is homogeneous of degree z, the value of $\widehat{F}$ on the unit sphere is given by*

$$\widehat{F}(\gamma) = \lim_{\epsilon \to 0} F(e^{-2\pi i \langle \gamma, \cdot \rangle} \eta_\epsilon) \tag{6}$$

where the convergence is uniform in γ. If, in addition, $\mathrm{Re}(z) > -Q - \min_j d_j$, *it is also true that*

$$\widehat{F}(\gamma) = \lim_{R \to \infty} \lim_{\epsilon \to 0} \left[\epsilon^{-Q-z} F(\eta) + \int K_{\epsilon,R}(\gamma,\omega) F(\omega) d\sigma(\omega) \right], \tag{7}$$

where $F(\omega)$ is the value of F on the unit sphere and the kernel $K_{\epsilon,R}$ is given by $K_{\epsilon,R}(\gamma,\omega) := \int_0^\infty e^{-2\pi i \langle \gamma, \rho \circ \omega \rangle} \rho^{Q+z-1} \left(\eta(R^{-1}\rho) - \eta(\epsilon^{-1}\rho) \right) d\rho$.

Proof. The distribution ηF is compactly supported, so its Fourier transform is equal to the C^∞ function $(\eta F)^\wedge(\xi) = F(e^{-2\pi i \langle \xi, \cdot \rangle}\eta)$. Notice that $\partial^\alpha F$ is homogeneous of degree $z - \alpha \cdot d$. For $|\alpha|$ sufficiently large, then, $\partial^\alpha F$ is integrable away from the origin. Therefore $\partial^\alpha((1-\eta)F) \in L^1(\mathbb{R}^n)$, so its Fourier transform, given by $(2\pi i \xi)^\alpha ((1-\eta)F)^\wedge$, is in $L^\infty(\mathbb{R}^n)$. It follows that $((1-\eta)F)^\wedge$ is of rapid decay away at infinity. It follows that, meaning that

$$\lim_{R \to \infty} \left[R^{Q+z} F(e^{-2\pi i \langle R \circ \gamma, \cdot \rangle}\eta) - R^{Q+z} \widehat{F}(R \circ \gamma) \right] = 0 \tag{8}$$

uniformly for all $\gamma \in S^{n-1}$. By homogeneity, (6) follows (replacing R^{-1} by ϵ).

To establish (7), observe that $\lim_{\epsilon\to 0}[\epsilon^{-w} F(e^{-2\pi i\langle \epsilon\circ\gamma,\cdot\rangle}\eta) - \epsilon^{-w}F(\eta)] = 0$ for any distribution F provided $\mathrm{Re}(w) > -\min_{j=1\ldots n} d_j$, since the test functions $\epsilon^{-\min d_j}(e^{-2\pi i\langle\epsilon\circ\gamma,\cdot\rangle} - 1)\eta(\cdot)$ have a limit in the Schwartz sense as $\epsilon \to 0$. If F is homogeneous of degree z for $\mathrm{Re}(z) > -Q - \min_j d_j$, it follows by scaling that $\lim_{\epsilon\to 0}\left[\epsilon^{-Q-z}F(\eta) - F(e^{-2\pi i\langle\gamma,\cdot\rangle}\eta_{\epsilon^{-1}})\right] = 0$. Inserting this limit into (8) gives

$$\widehat{F}(\gamma) = \lim_{R\to\infty}\lim_{\epsilon\to 0}\left[F(e^{-2\pi i\langle\gamma,\cdot\rangle}\eta_{R^{-1}}) - F(e^{-2\pi i\langle\gamma,\cdot\rangle}\eta_{\epsilon^{-1}}) + \epsilon^{-Q-z}F(\eta)\right];$$

but $\eta_{R^{-1}} - \eta_{\epsilon^{-1}}$ is compactly supported away from the origin, so F equals a function there and the equality (7) follows. □

3. Proof of Theorem 2. We turn now to the proof of theorem 2, which follows very closely the standard proof of the Plancherel theorem. The following lemma provides the identities necessary to proceed.

Lemma 2. *For any $z \in \mathcal{C}$ and any smooth functions f and g on S^{n-1},*

$$\int \mathcal{F}_z f(\gamma)g(\gamma)d\sigma(\gamma) = \int f(\gamma)\mathcal{F}_z g(\gamma)d\sigma(\gamma) \tag{9}$$

and

$$\mathcal{F}_{-Q-z}\mathcal{F}_z f(-\gamma) = f(\gamma). \tag{10}$$

Proof. To prove (9), it suffices by the uniform convergence of (6) to show that

$$\int E_z f(e^{-2\pi i\langle\gamma,\cdot\rangle}\eta_\epsilon)g(\gamma)d\sigma(\gamma) = \int f(\gamma)E_z g(e^{-2\pi i\langle\gamma,\cdot\rangle}\eta_\epsilon)d\sigma(\gamma) \tag{11}$$

for all $\epsilon > 0$. Both sides of (11) are meromorphic in z, so equality for all z follows from equality for $\mathrm{Re}(z) > -Q$. In this case $E_z f \in L^1_{loc}(\mathbb{R}^n)$, so the left-hand side of (11) is given by the integral

$$\iiint f(\gamma')g(\gamma)e^{-2\pi i\left\langle\gamma,\rho\circ\gamma'\right\rangle}\eta(\epsilon\rho)\rho^{Q+z-1}d\sigma(\gamma')d\rho d\sigma(\gamma). \tag{12}$$

Since (12) is symmetric in γ and γ', (11) follows by exchanging f and g.

To establish (10), note that $(E_z f)^\wedge = E_{-Q-z}(\mathcal{F}_z f)$ when $\mathrm{Re}(z) < 0$ since both distributions are equal on the unit sphere, homogeneous of the same degree, and locally integrable. Taking the Fourier transform of both sides and restricting to the unit sphere gives (10). Meromorphic continuation finishes the proof. □

Theorem 2 now follows from (9) and (10) after two elementary observations: $\overline{\mathcal{F}_z f(\gamma)} = \mathcal{F}_{\overline{z}} f^*(\gamma)$, where $f^*(\gamma) = \overline{f}(-\gamma)$, and $\overline{z} = -Q - z$. Using these identites gives the Plancherel identity immediately:

$$\begin{aligned}\int \mathcal{F}_z f(\gamma)\overline{\mathcal{F}_z f(\gamma)}\, d\sigma(\gamma) &= \int \mathcal{F}_z f(\gamma)\mathcal{F}_{\overline{z}} f^*(\gamma)\, d\sigma(\gamma)\\ &= \int \mathcal{F}_{\overline{z}}\mathcal{F}_z f(\gamma) f^*(\gamma)\, d\sigma(\gamma) = \int f(-\gamma)\overline{f}(-\gamma)\, d\sigma(\gamma)\\ &= \int |f(\gamma)|^2\, d\sigma(\gamma).\end{aligned}$$

4. Proof of Theorem 3. By (7), theorem 3 is a consequence of the following oscillatory integral estimate:

Lemma 3. *Given any real, distinct exponents $b_1, \ldots, b_m$, there is a constant C such that, for any polynomial $p(y) := \sum_{j=1}^m c_j y^{b_j}$ with real coefficients and any $z \in \mathbb{C}$ with $\mathrm{Re}(z) = 1$,*

$$\left| \int_0^\infty e^{ip(y)} \frac{dy}{y^z} \right| \leq C(|z-1|^{-\frac{1}{m+1}} + |z-1|^{-1}), \tag{13}$$

meaning that (13) holds uniformly when the endpoints of integration are taken to be $0 < A < B < \infty$.

For the moment, let us take the lemma for granted. For any positive ϵ and R, let $I(s) := \left\{ \rho \in (0,\infty) \;\middle|\; \eta(R^{-1}\rho) - \eta(\epsilon^{-1}\rho) > s \right\}$; by monotonicity of η, $I(s)$ is an interval. It follows that the kernel $K_{\epsilon,R}$ appearing in (7) satisfies (for $z = -Q + i\tau$) $K_{\epsilon,R}(\gamma,\omega) = \int_0^1 \int_{I(s)} e^{-2\pi i \langle \omega, \rho \circ \gamma \rangle} \rho^{-1+i\tau} d\rho ds$. By lemma 3, $|K_{\epsilon,R}(\gamma,\omega)| \leq C(|\tau|^{-\nu} + |\tau|^{-1})$, where $\nu^{-1} - 1$ equals the number of distinct exponents d_j. On the other hand, $E_z(\eta) = -\tau^{-1} \int \rho^{i\tau} \eta'(\rho) d\rho$ by (5); this has rapid decay as $|\tau| \to \infty$. Taking absolute values inside (7), it follows that $||\mathcal{F}_{-Q+i\tau} f||_\infty \leq C(|\tau|^{-\nu} + |\tau|^{-1})||f||_1$ as desired.

Proof of lemma 3. Without loss of generality, the exponents b_j may be taken to be non-zero. Let $z = 1 + i\tau$ for $\tau \neq 0$; for convenience, make the change of variables $y \mapsto e^x$. Just as in [5], the desired estimate has two main parts: the first is where the phase function $\Phi(x) = -\tau x + p(e^x)$ has small derivative; this set will be shown to be small. The second part is where the derivative is large; here van der Corput's lemma will be used.

Any function f which is a sum of exponentials is either identically zero or vanishes for no more than $m-1$ distinct values of x, where m is the number of exponential terms appearing in f. This claim can be established by induction: without loss of generality, $m \geq 1$ (otherwise $f \equiv 0$). Thus f has a term ce^{bx} with $c \neq 0$. Now $\frac{d}{dx}[e^{-bx} f]$ is a sum of fewer than m distinct exponentials, so is either identically zero or vanishes at no more than $m-2$ points. In either case, Rolle's theorem ensures that $e^{-bx} f(x)$ vanishes no more than $m-1$ times (since $f \not\equiv 0$); the same must then be true for f. In terms of $\Phi''(x)$, this means that one can partition the real line into m subintervals on which Φ' is monotone.

Consider the set $S_\epsilon := \{x \in R \mid |\Phi'(x)| < |\tau\epsilon| \}$. By monotonicity, this set is a union of m or fewer intervals. Let $I \subset S_\epsilon$ be any closed, bounded interval, and let $\delta := m^{-1}|I|$. Along the same lines as [5], define $D^s f(x) := e^{-sx} \frac{d}{dx}(e^{sx} f(x))$ for any $f \in C^\infty(\mathbb{R})$. It is elementary to check that $D^{-b_1} \cdots D^{-b_m} \Phi'(x) = (-1)^{m+1} \tau b_1 \cdots b_m$. Similarly, let $L^s f(x) := e^{s\delta} f(x+\delta) - f(x)$. In analogy with the mean value theorem, $L^s f(x) = s^{-1}(e^{s\delta} - 1) D_s f(x^*)$ for some $x^* \in [x, x+\delta]$. Since L^s and $D^{s'}$ commute, this equality may be iterated; it follows that $L^{-b_1} \cdots L^{-b_m} \Phi'(x) = -\tau \prod_{j=1}^m (e^{-b_j \delta} - 1)$. If x is taken to be the left endpoint of I, it must be the case that $|L^{-b_1} \cdots L^{-b_m} \Phi'(x)| \leq |\tau\epsilon| \prod_{j=1}^m (e^{-b_j \delta} + 1)$. For at least one j, then, $|e^{-b_j \delta} - 1| \leq |\epsilon|^{\frac{1}{m}} |e^{-b_j \delta} + 1|$; since $m\delta = |I|$, there exists a constant C depending only on m and the exponents b_j for which $|I| \leq C|\epsilon|^{\frac{1}{m}}$ whenever ϵ is sufficiently small. By monotonicity, the same is true of $|S_\epsilon|$.

The complement of $S_\epsilon \cap [A,B]$ can be partitioned into m or fewer intervals on which Φ' is monotone; on each, van der Corput's lemma is applied (as one has the bound $|\Phi'(x)| > |\tau\epsilon|$ here). Combining this with the estimate for $|S_\epsilon|$, the result is that $|\int_A^B e^{-i\tau x - ip(e^x)}dx| \leq m|\tau\epsilon|^{-1} + C|\epsilon|^{\frac{1}{m}}$. To obtain (13), fix ϵ when $|\tau|$ is small, and let $\epsilon = |\tau|^{-\frac{m}{m+1}}$ otherwise. $\square$

5. Functions Nearly Homogeneous at Infinity. Let $\mathcal{S}_z^\infty$ be the class of all C^∞ functions F for which $\sum_{j=1}^n d_j x_j \partial_j F - zF$ is a Schwartz function (for simplicity, define $\rho\partial_\rho := \sum_{j=1}^n d_j x_j \partial_j$). An $F \in \mathcal{S}_z^\infty$ will be called nearly homogeneous of degree z at infinity. In this section we prove several important properties of the Fourier transform of such functions F.

Lemma 4. *Let $F \in \mathcal{S}_z^\infty$, $G \in \mathcal{S}_w^\infty$. The following are true of F and G:*

1. *For any α, there exists $C_\alpha > 0$ such that $|\partial^\alpha F(x)| \leq C_\alpha (1+\rho(x))^{\mathrm{Re}(z) - \alpha\cdot d}$.*
2. *Away from the origin, $\widehat{F}$ is a C^∞ function of rapid decay.*
3. *If $\mathrm{Re}(z) < 0$, then $\widehat{F}$ is equal to an L^1 function.*
4. *If $\mathrm{Re}(z) > 0$, $\widehat{F}$ is a bounded linear functional on $C^N(\mathbb{R}^n)$ for large N.*
5. *$(\widehat{F})_\delta - \delta^{-Q-z}\widehat{F}$ is a Schwartz function depending smoothly on $\delta > 0$ (recall that $(\widehat{F})_\delta$ is the nonisotropic dilate of $\widehat{F}$).*
6. *Given any Schwartz function m, $|(m\widehat{F})^\vee(\xi)| \leq C(1+\rho(\xi))^{\mathrm{Re}(z)}$, where C depends continuously on m.*
7. *The product FG is in $\mathcal{S}_{z+w}^\infty$ and $(FG)^\wedge = \widehat{F} \star \widehat{G}$.*

Proof. Consider property 1; let $g(x) := -\int_1^\infty [\rho\partial_\rho F - zF](r \circ x) r^{-z-1} dr$. It follows that $\rho\partial_\rho g(x) - zg(x) = \rho\partial_\rho F(x) - zF(x)$ away from the origin, so it must be that the function $F(x) - g(x)$ is homogeneous of degree z on $\mathbb{R}^n \backslash \{0\}$. The function $g(x)$, however, is rapidly decaying as $\rho(x) \to \infty$, so F must satisfy the inequality $|F(x)| \leq C(1+\rho(x))^{\mathrm{Re}(z)}$ for some appropriate C. Furthermore, $\partial^\alpha(\rho\partial_\rho g) = \rho\partial_\rho(\partial^\alpha g) + (\alpha\cdot d)\partial^\alpha g$ for any smooth function g, so $\partial^\alpha F \in \mathcal{S}_{z-\alpha\cdot d}^\infty$, so it follows that F satisfies the differential inequalities $|\partial^\alpha F(x)| \leq C_\alpha(1 + \rho(x))^{\mathrm{Re}(z) - \alpha\cdot d}$ for all multi-indices α.

For property 2, the inequalities from property 1 are used as in the proof of lemma 1 to show that $\xi^\beta \partial_\xi^\alpha \widehat{F} \in L^\infty(\mathbb{R}^n)$ for $|\beta|$ sufficiently large because its Fourier transform, $[\partial_x^\beta(x^\alpha F)]^\wedge$ modulo constants, is in $L^1(\mathbb{R}^n)$.

Observe that $-\rho\partial_\rho \widehat{F} - (Q+z)\widehat{F} = (\rho\partial_\rho F - zF)^\wedge$, which is a Schwartz function. Taking $\widehat{g}(\xi) := -\int_1^\infty [\rho\partial_\rho \widehat{F} + (Q+z)\widehat{F}](r\circ\xi) r^{z+Q-1} dr$ as before, $\widehat{F}(\xi) - \widehat{g}(\xi)$ is homogeneous of degree $-Q-z$ away from the origin, and furthermore $|\widehat{g}(\xi)|$ is bounded by $C\rho(\xi)^{-Q-\mathrm{Re}\, z}$ if $\mathrm{Re}(z) > -Q$, by $C\ln(1/\rho(\xi))$ if $\mathrm{Re}(z) = -Q$, or by a constant if $\mathrm{Re}(z) < -Q$. It follows that $\widehat{F}$ is equal to an L^1 function away from the origin if $\mathrm{Re}(z) < 0$. Let Δ be the difference between $\widehat{F}$ and that L^1 function. Then $\Delta^\vee$ is a function that decays to 0 at infinity thanks to the decay of F and Riemann-Lebesgue. Since Δ is supported at the origin, $\Delta^\vee$ must equal a polynomial. It follows that $\Delta = 0$, proving property 3.

Now property 4: let $\widehat{F}_\alpha$ equal the distribution $\widehat{F}$ multiplied by x^α, and let $f(\alpha)$ equal the value of $\widehat{F}_\alpha$ tested on the function the constant function 1 (permissible because $\widehat{F}_\alpha$ decays rapidly away from the origin). Now $(\widehat{F}_\alpha)^\vee = (2\pi i \partial)^\alpha F \in s_{z-\alpha\cdot d}$ as already observed. If $\alpha \cdot d > \mathrm{Re}(z)$, property 3 dictates that $\widehat{F}_\alpha$ is an L^1 function. Taylor expand any test function φ up to degree N, with $N \min_j d_j > \mathrm{Re}(z)$, at the origin and test against $\widehat{F}$: $\widehat{F}(\varphi) = \sum_{|\alpha|<N} (\alpha!)^{-1} \partial^\alpha \varphi(0) f(\alpha) + \sum_{|\alpha|=N} (\alpha!)^{-1} \int \widehat{F}_\alpha(x) \varphi_\alpha(x) dx$, where $\varphi_\alpha(x)$ is a smooth, bounded function satisfying $|\varphi_\alpha(x)| \leq ||\partial^\alpha \varphi||_\infty$. Taking absolute values gives $|\widehat{F}(\varphi)| \leq ||\varphi||_{C^N}$.

Property 5 must hold because $(\widehat{F_\delta} - \delta^{-Q-z}\widehat{F})^\vee = F_{\delta^{-1}}(x) - \delta^{-z}F(x) = -\delta^{-z}\int_{\delta^{-1}}^1 [\rho\partial_\rho F - zF](r \circ x) r^{-z-1} dr$ which, by inspection, is a Schwartz function depending smoothly on δ.

To establish property 6, first observe that $(m\widehat{F})^\vee(\varphi) = F(\widehat{m} \star \varphi)$, from which it follows that $(m\widehat{F})^\vee$ is equal to the function $\int F(\xi+\nu)\widehat{m}(\xi) d\xi$. To estimate this integral let us show that, for any complex z, there is an N sufficiently large and a constant C' such that

$$\int (1+\rho(\xi+\nu))^{\mathrm{Re}(z)} (1+\rho(\xi))^{-N} d\xi \leq C'(1+\rho(\nu))^{\mathrm{Re}(z)}. \tag{14}$$

Suppose first that $\mathrm{Re}(z) \geq 0$. Recall that ρ is a norm. It follows that $(1+\rho(\xi+\nu))^{\mathrm{Re}(z)} \leq 2^{\mathrm{Re}(z)} \left[(1+\rho(\xi))^{\mathrm{Re}(z)} + \rho(\nu)^{\mathrm{Re}(z)}\right]$. Using this inequality to estimate the left-hand side of (14) gives the desired result, provided $N > Q - \mathrm{Re}(z)$ so that the integrals exist. Now suppose instead $\mathrm{Re}(z) \leq 0$. When $\rho(\xi) \leq \frac{1}{2}\rho(\nu)$, $\rho(\xi+\nu) \geq \frac{1}{2}\rho(\nu)$, so that $(1+\rho(\xi+\nu))^{\mathrm{Re}(z)} \leq 2^{-\mathrm{Re}(z)}(1+\rho(\nu))^{\mathrm{Re}(z)}$. In (14), use this inequality to estimate the left-hand side in the region where $\rho(\xi) \leq \frac{1}{2}\rho(\nu)$; otherwise use the trivial inequality $(1+\rho(\xi+\nu))^{\mathrm{Re}(z)} \leq 1$. When $N > Q$ it follows that the left-hand side of (14) is no greater than $C'\left[(1+\rho(\nu))^{\mathrm{Re}(z)} + (N-Q)^{-1}\rho(\nu)^{-N+Q}\right]$. Taking $N > Q - \mathrm{Re}(z)$ (again) gives (14). To finish, recall property 1, and observe $|\widehat{m}(\xi)| \leq C_N(1+\rho(\xi))^{-N}$ for all $N > 0$ since m is a Schwartz function (and the constants C_N may be taken to vary continuously with μ). These inequalities and (14) give property 6.

Finally, consider property 7. First observe that $\rho\partial_\rho(FG) - (z+w)FG = (\rho\partial_\rho F - zF)G + F(\rho\partial_\rho G - wG)$. Now $(\rho\partial_\rho F - zF)$ and $\rho\partial_\rho G - wG$ are Schwartz functions; the growth conditions on the derivatives of G provided by property 1 ensure that $(\rho\partial\rho F - zF)G$ is a Schwartz function (likewise for the remaining term), so $FG \in \mathcal{S}^\infty_{z+w}$. As for the Fourier transform, notice that the convolution $\widehat{F} \star \varphi(x) := \widehat{F}(\varphi(x - \cdot))$ is equal to the function $F(e^{-2\pi i\langle x, \cdot\rangle}\varphi^\vee)$, but, as before, this is a Schwartz function by virtue of property 1. Therefore $\widehat{F}$ may be convolved with any tempered distribution; since $(\widehat{F} \star \widehat{\varphi})^\vee(\xi) = F(\xi)\varphi(\xi)$, it will also hold that $(FG)^\wedge = \widehat{F} \star \widehat{G}$. $\square$

It should be noted that property 7 of lemma 4 is where, in some sense, the real usefulness of the classes $\mathcal{S}_z^\infty$ lies. For two homogeneous distributions, it may not always be the case that there exists another homogeneous distribution which is equal to their pointwise product away from the origin. If, however, we only require the distributions to be homogeneous in some asymptotic sense (as meant by the classes $\mathcal{S}_z^\infty$), one regains multiplicativity.

6. Nonisotropic Sobolev Spaces for the Unit Sphere. For each complex z, define the distribution $K^z := \eta_8[(1-\eta)\rho^z]^\vee$ (recall that the support of η_8 is on the nonisotropic ball of radius $\frac{1}{4}$ centered at the origin). Since $(1-\eta)\rho^z \in \mathcal{S}_z^\infty$, $[(1-\eta)\rho^z]^\vee$ differs from K^z by a Schwartz function; therefore $(K^z)^\wedge \in \mathcal{S}_z^\infty$ as well, and all the conclusions of lemma 4 hold for K^z.

Modulo a C^∞ function, $K^z \star g$ corresponds to the usual nonisotropic fractional derivative of g for any locally integrable function g. For any $z \in \mathbb{C}$, let $J^z f(\gamma) := K^z \star E_0 f(\gamma)$. Motiviated by the usual definition of Sobolev spaces via fractional differentiation, we make the following definition.

Definition. *We will say that the C^∞ function f on S^{n-1} is in L^p_s, for s real and non-negative, if $J^s f \in L^p$. This space is equuipped with the norm $||f||_{L^p_s} := ||f||_{L^p} + ||J^s f||_{L^p}$.*

An important remark is in order: one could instead define Sobolev spaces using the operator $J^s_w f(\gamma) := K^z \star E_w f(\gamma)$ for any $w \in \mathbb{C}$ (which is well-defined even when $E_w f$ does not exist as a distribution at the origin, because K^z has small support). This alternate choice produces the same space L^p_s—a result demonstrated by the following lemma. Throughout the remainder of this paper, it will be convenient to consider the following equivalence relation on operators. Two operators T_1 and T_2 acting on the space of functions on S^{n-1} will be considered equivalent, denoted $T_1 \sim T_2$, if there exists $K \in C^\infty(S^{n-1} \times S^{n-1})$ such that $T_1 f(\gamma) - T_2 f(\gamma) = \int K(\gamma, \omega) f(\omega) d\sigma(\omega)$.

Lemma 5. *The following are true of the fractional derivative operators J^s_w:*

1. $J^{z_2}_{w-z_1} J^{z_1}_w \sim J^{z_1+z_2}_{w_1}$.
2. *For any $s \leq t$ and any $w_1, w_2 \in \mathbb{C}$, there is a constant C such that $||J^s_{w_1} f||_p \leq C(||J^t_{w_2} f||_p + ||f||_p)$ for any smooth f. Consequently, the space L^p_s is independent of the degree of extension of f.*

Proof. Let $F(y) := K^{z_1} \star (E_{w_1} f)(y)$, which is well-defined for any complex w_1 provided y is in a neighborhood of the unit sphere. For δ near 1, $F_\delta(y) = \delta^{w_1} (K^{z_1})^{\delta^{-1}} \star (E_{w_1} f)(y) = [\delta^{w_1 - z_1} K^{z_1} - \delta^{w_1}(\delta^{-z_1} K^z - (K^z_1)^{\delta^{-1}})] \star (E_{w_1} f)(y)$. Near the unit sphere, then, $F_{\rho(y)^{-1}}(y) = E_0 J^{z_1}_{w_1}(y)$. By property 5 of lemma 4, it follows that,

$$E_0 J^{z_1}_{w_1} f(y) = \rho(y)^{z_1 - w_1} K^z \star (E_{w_1} f)(y) + \int S(y, \omega) f(\omega) d\sigma(\omega)$$

near the unit sphere, where S is some kernel depending smoothly on y and ω. Multiplying by $\rho(y)^{w_2}$ on both sides and convolving with K^{z_2}, it must be the case that, as operators acting on f,

$$J_{w_2}^{z_2} J_{w_1}^{z_1} f \sim [(m_\gamma K^{z_2}) \star K^{z_1}] \star (E_{w_1} f), \tag{15}$$

where $m_\gamma(x) := \rho(\gamma - x)^{z_1+w_2-w_1}$ (since, by properties 3 and 4 of lemma 4, convolving a smooth kernel with K^{z_2} gives a new smooth kernel). If $w_2 = w_1 - z_1$, it follows that $m_\gamma \equiv 1$ and property 1 must hold because $K^{z_2} \star K^{z_1}$ differs from $K^{z_2+z_1}$ by a compactly supported Schwartz function. Similarly, if $z_1 + z_2 = 0$, it follows that $J_{w_2}^{z_2} J_{w_1}^{z_1} f(\gamma) \sim f(\gamma)$.

Suppose that $-z_1 - z_2$ is a non-negative real number. Taking the Taylor series of m_γ at the origin, $m_\gamma K^{z_2} = \sum_{|\alpha|<N} (-1)^{|\alpha|} (\alpha!)^{-1} \partial^\alpha m_\gamma(0) K_\alpha^{z_2} + \sum_{|\alpha|=N} (-1)^{|\alpha|} (\alpha!)^{-1} m_\gamma^\alpha K_\alpha^{z_2}$, where $m_\gamma^\alpha(x)$ is a smooth function of γ and x and $K_\alpha^{z_2}$ is the distribution K^{z_2} multiplied by the function x^α. Recall $(K_\alpha^z)^\wedge \in \mathcal{S}^\infty_{z_2 - \alpha \cdot d}$, so $K_\alpha^{z_2} \star K^{z_1}$ is an L^1 function if $\alpha \neq 0$ by property 3 of lemma 4 (and also if $\alpha = 0$, provided $-z_1 - z_2 > 0$). Since m_γ^α is a Schwartz function when restricted to the support of $K_\alpha^{z_2}$, by property 6 of lemma 4, $(m_\gamma^\alpha K_\alpha^{z_2}) \star K^{z_1} \in L^\infty(\mathbb{R}^n)$ (and compactly supported) if $\alpha \cdot d > Q + z_1 + z_2$ (because $((m_\gamma^\alpha K^{z_2})^\wedge(\xi) K^{z_1}(\xi)^\wedge \in L^1(\mathbb{R}^n))$. It follows that

$$|[(m_\gamma K^{z_2} - K^{z_2}) \star K^{z_1}] \star (E_{w_1} f)(\gamma)| \leq C \int g(x) |E_{w_1} f(\gamma - x)| dx$$

for some constant C, where $g(x) := \eta_{\epsilon^4}(\rho(x)) + \sum_{0 \neq |\alpha| < N} |K_\alpha^{-z} \star K^z|(x)$. The function g is both integrable and supported in the nonisotropic ball of radius $\frac{1}{2}$ centered at the origin; by Minkowski's inequality, $||J_{w_2}^{z_2} J_{w_1}^{z_1} f||_p \leq ||J_{w_1}^{z_1+z_2} f||_p + C||f||_p$ for any $1 \leq p \leq \infty$ and some constant C independent of f. Since $z_1 + z_2$ is real and non-positive, it follows by lemma 4 that either $K^{z_1+z_2} \in L^1(\mathbb{R}^n)$ or $K^{z_1+z_2} - D \in L^1(\mathbb{R}^n)$ for the Dirac delta function D, so that $||J_{w_1}^{z_1+z_2} f||_p \leq C' ||f||_p$ as well.

To establish property 2, notice that $J_{w_1}^s \sim J_{w_1}^s J_{w_2-t}^{-t} J_{w_2}^t$ (again, properties 3 and 4 of lemma 4 ensure that the equivalence $\sim$ is preserved under composition with $J_{w_1}^s$). It was just shown that $J_{w_1}^s J_{w_2-t}^{-t}$ is bounded on L^p, so it must be the case that $||J_{w_1}^s f||_p \leq C(||J_{w_2}^t f||_p + ||f||_p)$. □

7. Proof of Theorem 4. The key to theorem 4 is the observation that $\mathcal{F}_w \mathcal{F}_z \sim J_z^{w+z+Q}$ for any $w, z \in \mathcal{C}$. To see this, fix smooth f on S^{n-1}. Let $E_w^\infty f(\rho \circ \gamma) := (1 - \eta(\rho)) \rho^w f(\gamma)$; by inspection $E_w^\infty f \in \mathcal{S}_w^\infty$. One may therefore define $\mathcal{F}_w^\infty f$ to be the value of $(E_w^\infty f)^\wedge$ on the unit sphere and $\mathcal{F}_w^0 f := \mathcal{F}_w f - \mathcal{F}_w^\infty f$. Consider first $\mathcal{F}_w^0$. If $\mathrm{Re}(w) > -Q$, $\mathcal{F}_w^0 f(\gamma) = \int S^w(\gamma, \omega) f(\omega) d\sigma(\omega)$, where $S^w(\gamma, \omega) := \int \eta(\rho) \rho^{w+Q-1} e^{-2\pi i \langle \gamma, \rho \circ \omega \rangle} d\rho$. Clearly $S^w(\gamma, \omega)$ is a smooth function of γ and ω; Integrating ρ^{w+Q-1} by parts repeatedly and continuing meromorphically shows that $\mathcal{F}_w^0 \sim 0$ for any $w \in \mathcal{C}$. Furthermore $\mathcal{F}_w^0 \mathcal{F}_z \sim 0$ for any $z \in \mathcal{C}$ since $\mathcal{F}_w^0 \mathcal{F}_z f(\gamma) = \int \mathcal{F}_z(S^w(\gamma, \cdot))(\omega) f(\omega) d\sigma(\omega)$ and $\mathcal{F}_z(S^w(\gamma, \cdot))(\omega)$ is smooth. Since $\mathcal{F}_w^0 \mathcal{F}_z \sim 0$, $\mathcal{F}_w^\infty \mathcal{F}_z f(\gamma) \sim [(1-\eta) \rho^{w+z+Q} (E_z f)^\wedge]^\vee(\gamma)$. Moreover $(1-\eta)\rho^{w+z+Q}$ differs from $(K^{z+w+Q})^\wedge$ by a Schwartz function, so $\mathcal{F}_w \mathcal{F}_z f(\gamma) \sim K^{z+w+Q} \star E_z f(\gamma) = J_z^{w+z+Q} f(\gamma)$.

Now fix $w \in \mathcal{C}$, $1 < p \leq 2$, and $t \geq s := \frac{Q}{p} + \mathrm{Re}(w)$. Under these conditions, both $-\frac{Q}{p} + i\,\mathrm{Im}(w)$ and $-Q - w + t$ are in $\mathcal{C}$. But

$$J^{-s+t}_{-Q-w+t}\mathcal{F}_w \sim \mathcal{F}_{-\frac{Q}{p}+i\,\mathrm{Im}(w)}\mathcal{F}_{-Q-w+t}\mathcal{F}_w \sim \mathcal{F}_{-\frac{Q}{p}+i\,\mathrm{Im}(w)}J^t_w,$$

so by theorem 1

$$||J^{-s+t}_{-Q-w+t}\mathcal{F}_w f||_{p'} \leq ||\mathcal{F}_{-\frac{Q}{p}+i\,\mathrm{Im}(w)}J^t_w f||_{p'} + C||f||_p \leq C'(||J^t_w f||_p + ||f||_p).$$

If $t = s$, $||\mathcal{F}_w f||_{p'} \leq C'(||J^s f||_p + ||f||_p)$ which is no greater than $C''(||J^t_w f||_p + ||f||_p)$ by property 2 of lemma 5. Restricting t to be non-negative, then, gives theorem 4. If $\mathrm{Im}(w) \neq 0$, the same argument is valid for $p = 1$.

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