Research Statement

Jaejeong Lee

My research interests are metric spaces of non-positive curvature. Under the guidance of my advisor Misha Kapovich, I had research experience in hyperbolic geometry and geometric group theory. I became interested in *convex* (*real*) *projective structures* (aka projective geometry on manifolds) because they serve as good models for studying the nature of non-positive curvature (as distinct from negative curvature in the sense of large-scale geometry). They are flexible enough to encompass many locally homogeneous geometric structures of non-positive curvature such as affine structures and hyperbolic structures. They carry an invariant metric called the Hilbert metric and it is natural to expect that the Hilbert metric somehow behaves like a non-positively curved metric.

A few examples and results are known in support of the above point of view [2, 3, 28, 35]. Certainly, however, there are not enough examples for one to conjecture any structure theorem. In the classical case of constant curvature metrics, it is well-known that every discrete group action has a convex fundamental polyhedron, namely, the *Dirichlet domain*. Thus every constant curvature manifold is obtainable from a convex polyhedron by gluing its facets by isometries. The *Poincaré fundamental polyhedron theorem* [20, 42] provides the necessary and sufficient conditions for this reverse construction to be possible.

The paucity of interesting examples in convex projective structure is mainly due to the lack of general fundamental domain theorems. In fact, even the existence of convex fundamental domains has been unknown. My advisor and I were naturally led to ask the following basic and complementary questions:

Question 1 (L-)**.** Does every convex projective structure admit a convex fundamental polyhedron?

Question 2 (Kapovich)**.** If so, does there exist a reverse process of constructing convex projective structures out of given convex polyhedra? In other words, is there a version of the Poincaré fundamental polyhedron theorem for convex projective structures?

In my dissertation research, I answered both questions in the affirmative – Theorem 3 and Theorem 5. Along the way, I developed my own point of viewing convex projective structures, which in turn gave me new insights on each locally homogeneous structures contained therein, e.g. hyperbolic structure – Proposition 4.

Convex projective structures

Let V be a real $(n+1)$-dimensional vector space. The projective n-sphere $\mathbb{S}(V)$ is the universal cover of the projective n-space $\mathbb{P}(V)$. The group of projective automorphisms of $\mathbb{S}(V)$ is isomorphic to the group $\mathrm{SL}^{\pm}(V)$ of real matrices of determinant ± 1. A point in $\mathbb{S}(V)$ represents a ray emanating from the origin in V. Thus a domain Ω in $\mathbb{S}(V)$ corresponds to an open cone Λ_Ω over Ω in V. The domain Ω is said to be convex (resp. properly convex) if the cone Λ_Ω is convex (resp. convex with no complete line).

A (*real*) *projective structure* is the geometric structure $(\mathbb{S}(V), \mathrm{SL}^{\pm}(V))$ modelled on the projective sphere with its automorphism group. Thus if M is a smooth n-dimensional manifold then a projective structure on M is equivalent to a pair of a local diffeomorphism $dev : \tilde{M} \to \mathbb{S}(V)$ and a homomorphism $\rho : \pi_1(M) \to \mathrm{SL}^{\pm}(V)$ such that dev is ρ-equivariant, that is, $dev \circ \gamma = \rho(\gamma) \circ dev$ for all $\gamma \in \pi_1(M)$ (see [22] and [23] for example). If the developing map dev is an embedding onto a convex (resp. properly convex) domain $\Omega \subset \mathbb{S}(V)$, then the projective structure on M is said to be *convex* (resp. *properly convex*).

Choi and Goldman classified all projective structures on compact surfaces (see [16] and references therein) – the deformation space of projective structures on a compact surface S is a countable disjoint union of open cells of dimension $-8\chi(S)$. In higher dimensions, however, not much is known of projective structures, although one remarkable fact in dimension 3 (Thurston, Molnár [41], Thiel [45]) is that if M admits one of the eight Thurston's geometries then either M or its double cover has a projective structure.

From now on, I will focus on (properly) convex projective structures on compact manifolds, which form a rather tractable subclass. To repeat the definition, a (properly) convex projective manifold M is the quotient $M = \Omega/\Gamma$, where $\Omega \subset \mathbb{S}(V)$ is a (properly) convex domain and $\Gamma \subset \mathrm{SL}^{\pm}(V)$ is a discrete subgroup acting on Ω freely, properly and cocompactly. Basic examples are those for which Ω is symmetric – every point of Ω is the only fixed point of a projective involution of Ω. Their classification follows from that of symmetric cones by Koecher [29, 46] and from the theory of lattices in semisimple Lie groups. These examples are uninteresting (in the viewpoint of the present statement). In view of Vinberg's result [47], the interesting examples are those for which Ω is non-homogeneous.

The first non-homogeneous example was found by Kac and Vinberg [49]. More non-homogeneous examples in small dimensions were obtained from the Tits–Vinberg fundamental domain theorem [6, 48] for discrete linear groups generated by reflections. This constrution, however, does not apply in higher dimensions [26]. Later on, using Koszul's stability theorem [30], Benoist [1] constructed non-homogeneous examples in every dimension via projective bending deformation [24, 27] of cocompact lattices in $\mathrm{O}^+(n,1)$. Non-homogeneous examples for which Γ is not isomorphic to a cocompact lattice in $\mathrm{O}^+(n,1)$ were constructed by Benoist [3, 4] in small dimensions and by Kapovich [28] in all dimensions ≥ 4. See [5] for more detailed overview on the subject.

Dissertation work toward Question 1

Recall that V denotes a real $(n+1)$-dimensional vector space. Suppose that Ω is a properly convex domain in $\mathbb{S}(V)$ and $\Gamma \subset \mathrm{SL}^{\pm}(V)$ is a discrete subgroup acting on Ω freely and properly discontinuously. Then, by definition, the quotient Ω/Γ is a properly convex projective n-manifold. Question 1 at the beginning can be rephrased as: "Does the action of Γ on Ω have a convex fundamental polyhedron?"

In order to study the geometry of the domain Ω, one usually considers the Hilbert metric d_Ω on Ω which is defined in terms of cross ratio (see [9] or [23] for example). It is a complete proper metric invariant under $\mathrm{Aut}(\Omega) \subset \mathrm{SL}^{\pm}(V)$ and, when the domain Ω is an ellipsoid, it coincides with the hyperbolic metric – the metric of constant curvature -1. One disadvantage of using the Hilbert metric d_Ω, however, is that the bisector of two points with respect to d_Ω is not necessarily totally geodesic. In fact, Busemann [8, Theorem 47.4] showed that if every bisector is totally geodesic then the metric space (Ω, d_Ω) must be isometric to either $\mathbb{E}^n$ or $\mathbb{H}^n$. Therefore, in general, the Dirichlet fundamental domain of Γ with respect to the Hilbert metric cannot be convex. For this reason, it has been unknown to researchers in the field whether the action of Γ on Ω has a convex fundamental domain.

Recently, I proved the following theorem thereby answering Question 1 affirmatively in the case of properly convex projective structures:

Theorem 3 (L-, [33])**.** *Every properly convex projective structure has a convex fundamental polyhedron.*

Instead of using the Hilbert metric on $\Omega \subset \mathbb{S}(V)$, I worked in the affine space V and used the correspondence between properly convex projective structures and the affine differential geometry of hypersurfaces; this correspondence was first observed by Labourie [31, 32] and Loftin [38] independently. Namely, one considers the convex cone Λ_Ω over Ω in V. The group $\Gamma \subset \mathrm{SL}^{\pm}(V)$ still acts on this cone. In this context, Calabi [10] conjectured that there is a strictly convex complete hypersurface M (called *hyperbolic affine sphere*) which is asymptotic to the boundary of Λ_Ω and which is invariant under the action of Γ. Calabi's conjecture was proved by Cheng and Yau [13, 14] (see also [43], [21] and [36, 37]).

My idea of proving Theorem 3 was to take a point $x_0 \in M$ and to consider the intersection

$$E(x_0) = \bigcap_{g\in\Gamma} H^+(gx_0)$$

of halfspaces bounded by those hyperplanes which are tangent to M at the Γ-orbit of x_0. Because M is asymptotic to the boundary of Λ_Ω, these tangent hyperplanes are locally finite in Λ_Ω and hence $E(x_0)$ is a polyhedron invariant under Γ. Because M is strictly convex, each tangent hyperplane defines a facet of $E(x_0)$, which is a convex polyhedron of dimension n. Via projectivization into Ω, each facet of $E(x_0)$ gives rise to a fundamental polyhedron for the action of Γ on Ω. This completes the proof.

I also studied the relation between the polyhedron $E(x_0)$ defined above and the convex hull $I(x_0) = \mathrm{conv}\{gx_0 \,|\, g \in \Gamma\}$ of the Γ-orbit of $x_0 \in M$. I proved

Proposition 4 (L-, [33]). *If Ω is a symmetric domain, then the tessellations of Ω obtained by projectivizing the facets of $E(x_0)$ and $I(x_0)$ are Poincaré dual to each other.*

In particular, the above result holds when Ω is an ellipsoid – the Klein model of hyperbolic n-space. Proposition 4 seems to be new even in this special case[1].

My idea of proof was to interpret the so-called conormal duality between hyperbolic affine spheres in terms of the linear duality between certain convex cones in $V \times \mathbb{R}$ defined as follows. Consider the dual cone Λ^*_Ω of Λ_Ω, which determines a hyperbolic affine sphere M^* according to Calabi's conjecture. There is a natural map (called *conormal* map) $x \mapsto x^*$ between M and M^*, which is equivariant under the isomorphism $g \mapsto (g^t)^{-1}$ from Γ to Γ^t, where t denotes taking the transpose. Let $I(x^*_0)$ denote the convex hull of the Γ^t-orbit of $x^*_0 \in M^*$. I showed that the cones over $E(x_0) \times \{\pm 1\}$ and $I(x^*_0) \times \{\pm 1\}$ in $V \times \mathbb{R}$ are polyhedral cones which are dual to each other in the usual sense. Thus the polyhedron $E(x_0)$ is dual to $I(x^*_0)$. When $\Omega = \Omega^*$ is symmetric, the conormal map $x \mapsto x^*$ is an isometry. Therefore, $I(x_0)$ and $I(x^*_0)$ are isomorphic and this proves Proposition 4. In general, one can consider the cones over $M \times \{\pm 1\}$ and $M^* \times \{\pm 1\}$ in $V \times \mathbb{R}$, which I call *extension cones*. This is exactly the reverse process of obtaining a hyperbola as a conic section and is analogous to the totally geodesic embedding of $\mathbb{H}^n$ into $\mathbb{H}^{n+1}$.

Finally, I interpreted the fundamental domain in Theorem 3 obtained by projectivizing a facet of $E(x_0)$, as a Dirichlet domain with respect to the invariant two-variable function $\mathcal{S}$ on M defined by

$$\mathcal{S}(x, y) = \langle x^*, y \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on V. When the domain Ω is the symmetric space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$, the cone Λ_Ω over Ω is the cone of positive-definite symmetric $n \times n$ matrices. In this case, the function $\mathcal{S}$ coincides with the function

$$d(x, y) = \mathrm{tr}(x^{-1}y)[\det(xy^{-1})]^{1/n},$$

which Selberg introduced in his paper [44] to construct convex fundamental polyhedra of lattices in $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$. Selberg gave no explanation of how he derived the function d. In fact, my work was initiated from my attempt to understand this function in the general case when Ω is an arbitrary properly convex domain.

Dissertation work toward Question 2

Now that we know the existence of convex fundamental polyhedron, it is natural to seek for a version of Poincaré fundamental polyhedron theorem for convex projective

[1] Recently, I was informed that this special case of Proposition 4 was proved in [12, Lemma 5].

structures. The non-trivial but elementary issue here is that the union of convex sets is not convex in general; what Question 2 really asks for is a convexity criterion for given projective structure.

As mentioned at the beginning, there have been only a few methods for constructing convex projective structures – each with its own limited applicability. In dimension 2, Goldman [24] made use of the pants decomposition of surfaces. As explained before, the Tits–Vinberg fundamental domain theorem is applicable in small dimensions only; see [3, 4] and [15] for examples constructed by this method. Modifying the Tits–Vinberg construction, Kapovich proved a convexity theorem [28, Theorem 4.3] which applies to infinite-sided polyhedra. Although the theorem of Kapovich is more or less specialized to his own purpose on Gromov-Thurston manifolds [25], it also shows how useful it can be to have a convexity theorem in greater generality.

My version of a convexity theorem for projective structures is as follows:

Theorem 5 (L-, [34])**.** *Given a finite collection $\mathcal{P}$ of convex n-polytopes in $\mathbb{S}(V)$, suppose that M is a projective manifold which is obtained by gluing together the polytopes in $\mathcal{P}$ along their facets in such a way that the union of any two adjacent polytopes sharing a common facet is convex. Then, the projective structure on M is*

1. *convex if $\mathcal{P}$ contains no triangular polytope, and*
2. *properly convex if, in addition, $\mathcal{P}$ contains a polytope whose dual polytope is thick.*

The condition that the union of any two adjacent polytopes is convex is called the *residual convexity* condition in [34] and was first introduced in [28]. It is a local condition and thus Theorem 5 can be regarded as a local-to-global theorem on convexity; for this part of argument, I put the standard Riemannian metric on $\mathbb{S}(V)$ and utilized the local-to-global theorem [7] for Alexandrov spaces of curvature ≥ 1.

Triangular polytopes and polytopes with thick duals are defined as analogues of triangles and polygons with at least five edges, respectively. As a matter of fact, my research toward the above theorem was initiated by observing its validity in dimension 2 under the no-triangle condition and under the existence of a polygon with at least five edges, respectively. After defining the analogues of these notions appropriately, I was able to go from dimension two to arbitrary dimensions using the *codimension-*2 *phenomena* in polyhedral complexes, which is common, for example, in the proof of the classical Poincaré fundamental polyhedron theorem [20, 42].

While Theorem 5 is limited to polytopes, i.e. compact polyhedra, it does not require that the gluing maps be reflections and it is valid in all dimensions. Furthermore, because gluing by reflections necessarily gives rise to residually convex tessellations, Theorem 5 does generalize the Tits–Vinberg fundamental domain theorem.

Future research plan

Affine structures are convex but not properly convex projective structures. The condition of proper convexity in Theorem 3 remains open the following problem:

Question 6 (Kapovich)**.** Does every compact affine structure have a convex fundamental polyhedron?

The Margulis' examples [39, 40] of non-compact affine manifold with non-abelian free fundamental group do not seem to have a convex fundamental domain [19], hence the compactness assumption in Question 6. See the survey paper [11] and references therein for non-compact affine manifolds and their fundamental domains. Compact complete affine manifolds are classified in small dimensions only. Thus a positive answer for Question 6 might help their classification in higher dimensions. I will investigate affine structures and try the above question, because the study of affine structure is also necessary for approaching Question 8 below.

Question 7 (L-)**.** Does the conclusion of Proposition 4 hold for non-homogeneous domain Ω?

For any properly convex cone C, one may consider the characteristic function ϕ_C of C (see [46] for example). When C is homogeneous, it is known [43] that the level surfaces of ϕ_C coincide exactly with the hyperbolic affine spheres associated to C. While some of the nice properties of ϕ_C hold only if C is homogeneous, the corresponding properties are still valid for hyperbolic affine spheres regardless of homogeneity of cones. Therefore, in order to study non-homogeneous cones (and domains), it is desirable to consider hyperbolic affine spheres. In this regard, Question 7 seems to be one of the basic questions to be answered. I have investigated this problem and will continue my work on it.

Having obtained Theorem 5, I plan to construct explicit examples. I want to do so by investigating the following outstanding problems in the subject:

Question 8 (Thurston)**.** Do we have a toral combination theorem?

To explain the question, let M be a compact 3-manifold with toral boundary and let DM denote the double of M. Suppose that the interior of M admits a hyperbolic structure. Question 8 asks, for example, if the doubled manifold DM admits a convex projective structure. Benoist [3] constructed a few such examples. I want to investigate the deformation space of cusped convex projective manifolds to approach the above question. As a related problem, Cooper, Hodgson and Kerckhoff have been investigating hyperbolic Dehn surgery in the context of projective structures.

Question 9 (Cooper-Long-Thistlethwaite, [17, 18])**.** What is the necessary and sufficient conditions for a closed hyperbolic 3-manifold to flex?

Cooper, Long and Thistlethwaite have found some interesting examples of closed non-Haken hyperbolic 3-manifolds which admit non-trivial deformations into convex projective structures. They called this phenomena *flexing*. According to their expression, Question 9 and how flexing occurs seem very mysterious. I wonder if the flexing phenomena has something to do with polyhedral rigidity of the fundamental polyhedron. I plan to investigate the fundamental polyhedra of flexible manifolds.

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