

Research Statement

Jayadev S. Athreya

November 1, 2008

1 Introduction

My research interests lie at the intersection of dynamics, geometry, and the theory of Lie groups. More precisely, my research up to this point has two main themes:

- Deriving *quantitative* information about recurrence properties of flows on moduli and parameter spaces of geometric objects, and the applications of these results.
- Using mixing and ergodic properties of those flows to derive results on counting problems.

The flows I consider are mainly actions of one-parameter subgroups of certain Lie groups. Much of my work is inspired by comparing the properties of actions on homogeneous spaces G/Γ (G a Lie group, Γ a discrete subgroup of G), and the dynamics of the $SL(2, \mathbb{R})$ action on the moduli space of quadratic differentials.

2 Billiards and Teichmüller dynamics

The connection between dynamics of billiard flows in polygons (and more generally, of straight-line flows on flat surfaces) and Teichmüller dynamics has been the subject of much recent study. Many of my research projects have focused on developing these connections, both for ergodic properties and counting problems for billiards in polygons.

2.1 Deviation for ergodic averages for billiards in polygons

In joint work with Giovanni Forni [5], we study the deviation of ergodic averages for billiard flows: let $P \subset \mathbb{R}^2$ be a Euclidean polygon and let $G(P) \subset O(2, \mathbb{R})$ be the subgroup generated by all reflections with axis parallel to an edge of P (and passing through the origin). The polygon P is called *rational* if $G(P)$ is finite. A necessary condition, which is also sufficient if P is simply connected, is that the angles of P belong to $\pi\mathbb{Q}$. The billiard flow on P is a discontinuous Lagrangian (Hamiltonian) flow on the unit tangent bundle $T_1(P) \equiv P \times S^1$.

The trajectory of any $(x, v) \in T_1(P)$ moves with unit speed along a straight line in the direction $v \in S^1$ up to the boundary ∂P where it is reflected according to the law of geometric optics (*angle of incidence equal angle of reflection*), which follows from the assumption that collisions with the boundary are elastic.

The billiard flow on a rational table P leaves invariant the angle function $\Theta : P \times S^1 \rightarrow \mathbb{R}$ obtained as a composition of the canonical projections

$$\Theta : P \times S^1 \rightarrow S^1 \rightarrow S^1/G(P)$$

onto the quotient $S^1/G(P)$, which can be identified to a compact interval $I(P) \subset \mathbb{R}$. It follows that the phase space $P \times S^1$ is foliated by the level surfaces $S_{P,\theta} = \{(x, v) \in P \times S^1 \mid \Theta(x, v) = \theta\}$ which are invariant under the billiard flow. If P is rational billiard table, let $A_{P,\theta}$ denote the area form on the invariant translation surface $S_{P,\theta}$ for all $\theta \in S^1$. We have the following result:

Theorem 1. *Let $P \subset \mathbb{R}^2$ be a rational polygon. For any $\theta \in S^1$, let $\psi_{t,\theta}$ be the restriction of the billiard flow to the invariant surface $S_{P,\theta} \subset P \times S^1$. There exist an $\alpha = \alpha(P) > 0$ (depending only on the shape of P , in particular on the stratum arising from the unfolding procedure) and a measurable function $K_P : S^1 \rightarrow \mathbb{R}^+$ such that, for almost all $\theta \in S^1$, for all f in the standard Sobolev space $H^1(S_{P,\theta})$, and all $x \in S_{P,\theta}$ for which $\psi_{\theta,t}(x)$ is defined for all $t > 0$,*

$$\left| \int_0^T f(\psi_{\theta,t}(x)) dt - T \int f dA_{P,\theta} \right| \leq K_P(\theta) \|f\|_{H^1(S_{P,\theta})} T^{1-\alpha}. \quad (1)$$

The key tools in the proof of this result are Forni's estimates [14] on the growth of the Kontsevich-Zorich cocycle, and the quantitative recurrence results of my thesis [1], which we describe in subsection 2.4 below.

2.2 Lattice point asymptotics and volume growth in Teichmüller space

In joint work with Sasha Bufetov, Alex Eskin (my thesis advisor), and Maryam Mirzakhani [3], we studied some asymptotic properties of Teichmüller space.

Let $g \geq 2$ and let Q_g be the Teichmüller space of unit-area quadratic differentials on marked compact Riemann surfaces of genus g , and let Γ be the mapping class group. The space Q_g carries a natural normalized smooth measure μ , preserved by the action of Γ and such that $\mu(Q_g/\Gamma) < \infty$ (Masur [19], Veech [25]).

Let T_g be the Teichmüller space of marked compact Riemann surfaces of genus g . We have a natural projection

$$\pi : Q_g \rightarrow T_g$$

and we set $m = \pi_*\mu$.

Now we take $x, y \in T_g$ and we let $B_R(x)$ be the ball of radius R in the Teichmüller space in the Teichmüller metric, centered at the point x .

Our main results are the following two theorems:

Theorem 2 (Lattice Point Asymptotics). *As $R \rightarrow \infty$,*

$$|\Gamma y \cap B_R(x)| \sim \frac{1}{h} \Lambda(x) \Lambda(y) e^{hR},$$

where $h = 6g - 6$ is the entropy of the Teichmüller geodesic flow, and $\Lambda : T_g \rightarrow \mathbb{R}$ (the Hubbard-Masur function) is a bounded, Γ -invariant function.

Theorem 3 (Volume Asymptotics). *As $R \rightarrow \infty$,*

$$m(B_R(x)) \sim \frac{1}{h} e^{hR} \Lambda(x) \cdot \int_{T_g/\Gamma} \Lambda(y) dm(y).$$

These results are directly analogous to those proved for compact manifolds of negative curvature (in particular hyperbolic manifolds) in the Ph.D. thesis of G. A. Margulis. It is one part of a fruitful program to generalize other results on the geometry and dynamics of hyperbolic manifolds to Teichmüller space. There are many open questions in this direction as well:

For example, in [21], Maucourant proved the following shrinking target and logarithm law result (see section 3 for more results of this type) for geodesic flows on hyperbolic manifolds:

Theorem 4. [21] *Let $\Gamma \subset SO(k+1, 1)$ be a non-uniform lattice, $x, y \in X = \mathbb{H}^{k+1}/\Gamma$. Let $\{g_t\}$ denote the one parameter subgroup corresponding to geodesic flow on the unit tangent bundle SX , and $\pi : SX \rightarrow X$ the projection map. Let $\{B_t\}_{t \geq 0}$ be a decreasing family of closed metric balls of radius $\{r_t\}_{t \geq 0}$. Then for all $x \in X$,*

$$\lambda\{\theta \in S_x X : \{t \geq 0 : \pi(g_t(x, \theta)) \in B_t\} \text{ is unbounded}\} = \begin{cases} 1 & \int_{t=0}^{\infty} r_t^k = \infty \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

As a corollary, one obtains a logarithm law for visits to balls: for all $x, y \in X$, almost every $\theta \in S_x X$,

$$\limsup_{t \rightarrow \infty} \frac{-\log d(\pi(g_t(x, \theta)), y)}{\log t} = 1/k.$$

It would be of great interest to obtain an analog of this result for Teichmüller geodesic flow (see section 2.4 for details). The key technique in Maucourant's work is an L^2 -estimate that relies on two things:

- An exponential estimate for the lattice point problem.
- Two-point homogeneity of hyperbolic manifolds.

Theorem 2 yields the first requirement. However, Teichmüller space is not even one-point homogeneous. Despite this difficulty, I believe that there is a 'measurable' analogue of this property (which would be of independent interest) that should allow one to derive the required estimate.

2.3 Rectangular Billiards

In joint work with Alex Eskin and Anton Zorich [4], we obtain asymptotics for the number of closed trajectories for billiards in tables with right angles. There are very few counting results for billiard flows on polygons, since, as discussed above, the set of surfaces arising from billiards in any given stratum is of measure 0.

Precisely, we have the following: let S denote a billiard table with all angles multiples of $\pi/2$, and let $N(S, r)$ denote the number of closed billiard trajectories on S of length less than or equal to r .

Theorem 5. *There is a constant c depending only on the shape of the table and a function $e(R)$ such that for almost every rectangular billiard S ,*

$$N(S, R) = cR^2 + e(S, R),$$

where

$$\lim_{R \rightarrow \infty} \frac{1}{\log R} \int_1^R \frac{e(S, r)}{r^3} dr = 0.$$

The proof of this result follows techniques developed by Eskin and Masur in [12], which relate the counting problem to a problem in the ergodic theory of the Teichmüller flow on a certain subset of Q_g , for appropriate g . In this case, $g = 0$, and the subset is a collection of meromorphic quadratic differentials with simple poles.

In order to calculate the constants, one must calculate volumes of certain moduli spaces. For this, we follow techniques of Eskin and Okounkov [13], and use a generalization of a counting formula of Kontsevich [17].

2.4 Quantitative recurrence for Teichmüller flow

Theorem 1 on deviation for ergodic averages can be viewed as a quantitative strengthening of the celebrated result of S. Kerckhoff, H. Masur and J. Smillie [15] on the unique ergodicity for the directional flow of a rational polygonal billiard in almost all directions.

The key technical tool in their paper is a non-divergence result for the Teichmüller flow. With notation as in section 2.2, Q_g (and Q_g/Γ) can be stratified according to the orders of the zeros of the quadratic differentials (i.e., by integer partitions of $2g - 2$).

There is a natural $SL(2, \mathbb{R})$ action on Q_g which preserves μ , and in fact preserves a natural measure on each stratum \mathcal{H} of Q_g/Γ . We are particularly interested in the action of the diagonal subgroup $A = \left\{ g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}$. This action is known as the *Teichmüller geodesic flow*. Other subgroups of interest are the circle group $K = \left\{ r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} : 0 \leq \theta < 2\pi \right\}$ (circle flow) and $N = \left\{ h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$ (horocycle flow). The geodesic and

horocycle flows are known to be ergodic and even mixing with respect to the measure μ , by results of Masur [19] and Veech [25].

Masur [20] showed that if $q \in Q_g$, and $\{g_t q\}_{t \geq 0}$ is non-divergent (i.e., makes infinite returns to *some* compact set), the vertical flow on the (half)-translation surface associated to q is uniquely ergodic. Combining this with the ergodicity of $\{g_t\}$, we obtain that for almost every $q \in Q_g$, the vertical flow is uniquely ergodic. However, the set of $q \in Q_g$ arising from unfolding of billiards is a set of μ -measure 0, so this result does not yield information about billiards.

However, if we consider our polygon P , the invariant surfaces $S_{P,\theta}$ for the billiard flow have natural translation structures induced by the translation structure on $P \subset \mathbb{R}^2$ (and thus can be viewed as points in $q_{P,\theta} \in Q_g$ for some $g \geq 1$, where Q_g is the moduli space of quadratic differentials, with the flow on the billiard in direction θ being associated to the *vertical* flow associated to $q_{P,\theta}$). By following the unfolding construction of Zemljakov-Katok [26], it is possible to show that all invariant translation surfaces $S_{P,\theta}$ for $\theta \notin \Theta^{-1}(\partial I(P))$ can be identified with rotations of a fixed translation surface S_P of genus $g(P) \geq 1$ so that the billiard flow restricted to $S_{P,\theta}$ can be identified with the directional flow on S_P in the direction at angle $\theta \in S^1$ from the vertical.

Using this construction, Kerckhoff-Masur-Smillie circumvented the difficulty of billiards being a set of measure 0 by showing that for all $q \in Q_g$, and almost every $\theta \in S^1$, $\{g_t r_\theta q\}_{t \geq 0}$ is non-divergent, and thus, for all (half)-translation surfaces (in particular those arising from billiards), the vertical flow in almost every direction is uniquely ergodic.

The result Theorem 1 can be viewed as a quantitative strengthening of the unique ergodicity result, and it relies on a quantitative strengthening of the corresponding recurrence result:

Theorem 6. *Let $q \in Q_g$. There is a compact set C and constants $c_1, c_2 > 0$ and $\lambda < 1$ such that*

$$|\{\theta : \frac{1}{T}|\{0 \leq t \leq T : g_t r_\theta q \in C\}| < 1 - \lambda\}| \leq c_1 e^{-c_2 T},$$

for all T sufficiently large. Moreover, given any $\lambda < 1$, there is a compact set $C = C_\lambda$ such that for almost all $\theta \in S^1$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} |\{0 \leq t \leq T : g_t r_\theta q \in C\}| > 1 - \lambda.$$

In intuitive terms, this result states that for almost all quadratic differentials $q \in Q_g$, if we pick a rotation $r_\theta q$ at random (according to Lebesgue measure on the circle $[0, 2\pi)$), the likelihood of picking a trajectory that has spent a large proportion of its life up to time T outside a compact set decays exponentially in T , and that given any desired proportion, we can pick a compact set so that in almost every direction, our Teichmüller geodesic trajectory spends (eventually) that proportion of its time in that compact set. This result is very similar in spirit to those of Minsky-Weiss [22] on non-divergence of the horocycle flow.

While ergodicity guarantees that $\frac{1}{T}|\{0 \leq t \leq T : g_t q \in C\}| \rightarrow \nu(C)$ for μ -almost every $q \in Q_g(P)$, our result gives explicit information about the likelihood of bad trajectories.

In [11], Avila, Gouezel, and Yoccoz use similar results to show that the rate of mixing for the geodesic flow is exponential, making it one of the first examples of an exponentially mixing flow outside of surfaces of negative curvature. This also yields the existence of a spectral gap for the $SL(2, \mathbb{R})$ action on Q_g .

3 Logarithm laws and shrinking target properties

A classical problem in ergodic theory and dynamical systems is as follows: given an ergodic measure preserving system $T : (X, \mu) \rightarrow (X, \mu)$, and a sequence of (measurable) ‘target’ sets $A_n \subset X$, we study the set $A_\infty = \{x \in X : T^n x \in A_n \text{ infinitely often}\}$, usually seeking to prove a 0 – 1 law. Generally, we have $\mu(A_n) \rightarrow 0$, giving rise to the term ‘shrinking target’. Of particular interest is when X is a non-compact space, and the sets A_n are neighborhoods of infinity. Often times the excursions to these neighborhoods will follow a logarithmic pattern. Many of our recent results can be viewed as examples of this phenomenon. We have written a survey [2] summarizing some of the recent progress in these directions, and we describe some of our results below.

3.1 Unipotent logarithm laws

Another interesting example of a parameter space of geometric objects is the space of X_n of unimodular lattices in \mathbb{R}^n . X_n can be naturally identified with $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$. X_n is non-compact, but has a natural finite volume coming from Haar measure on $SL(n, \mathbb{R})$. Let $\mu = \mu_n$ be this measure on X_n , normalized to be a probability measure. Define $\alpha_1 : X_n \rightarrow \mathbb{R}^+$ by

$$\alpha_1(\Lambda) := \sup_{0 \neq v \in \Lambda} \frac{1}{\|v\|}.$$

Mahler’s compactness criterion states that α_1 is a proper function on X_n , that is, it is unbounded off compact sets.

Given a non-compact one-parameter subgroup $\{\phi_t\}_{t \in \mathbb{R}} \in G$, the action on X_n is ergodic with respect to μ . In particular, for μ -almost every $\Lambda \in X_n$, $\{\phi_t X\}_{t \geq 0}$ is dense in X_n . A natural question is to understand the behavior of excursions of these trajectories into the cusp of X_n . For diagonal (i.e., Cartan) subgroups $\{g_t\}$, Kleinbock-Margulis [16], generalizing work of Sullivan [23] on hyperbolic manifolds, proved the following result:

Theorem 7. *For μ - a.e. $\Lambda \in X_n$,*

$$\limsup_{t \rightarrow \infty} \frac{\log \alpha_1(g_t \Lambda)}{\log t} = \frac{1}{n}.$$

In joint work with G. Margulis [8], we prove the following analogous result for *unipotent* flows. Let $\{u_t\}_{t \in \mathbb{R}}$ denote a unipotent one-parameter subgroup of $SL(n, \mathbb{R})$.

Theorem 8. *For μ - a.e. $\Lambda \in X_n$,*

$$\limsup_{t \rightarrow \infty} \frac{\log \alpha_1(u_t \Lambda)}{\log t} = \frac{1}{n}.$$

It is somewhat surprising that while these two flows behave in very different ways in many respects (in particular their rate of mixing) they share this statistical behavior of excursions to cusps. It would also be interesting to study, in analogy with Theorem 4, the statistical behavior of the returns of these flows to shrinking balls.

Unsurprisingly, the proofs of these two theorems are significantly different: Kleinbock-Margulis use the exponential rate of mixing of Cartan flows, while our main tool in the proof of Theorem 8 is a measurable analogue to Minkowski's convex body theorem, which is of independent interest. Recall that Minkowski's theorem states that if $A \subset \mathbb{R}^n$ is a convex, centrally symmetric region with $m(A) > 2^n$ (m is Lebesgue measure on \mathbb{R}^n), then for all $\Lambda \in X_n$, there is a non-zero vector $v \in \Lambda \cap A$.

Without the strong assumptions on the geometry of A , the result fails. However, one can ask a measure question: given a set A of large measure, what is the measure (in X_n) of the set of lattices that do not intersect A ?

Theorem 9. *Let $n \geq 2$. There is a constant C_n such that if A is a bounded Borel measurable set in \mathbb{R}^n , with $m(A) = a > 0$,*

$$\mu(\Lambda \in X_n : \Lambda \cap A = \emptyset) \leq \frac{C_n}{a}$$

As an example of the back-and-forth between dynamics on G/Γ and Teichmüller dynamics that is present in my research, I have also proved an analogous result for the moduli space of quadratic differentials in joint work with Yair Minsky [9]. Let \mathcal{H} be a stratum of Q_g/Γ (with notation as in section 2.4).

Similar to the space of lattices, each stratum is endowed with a proper function $\lambda : \mathcal{H} \rightarrow \mathbb{R}^+$ be defined by

$$\lambda(q) = \sup_{v \in V_{sc}(q)} \frac{1}{\|v\|},$$

where $V_{sc}(q)$ is the set of (holonomy vectors of) saddle connections on q . Recall that a *saddle connection* on a q is a geodesic (in the flat metric defined by q) connecting two zeros of q with no zeros in its interior).

Theorem 10. *Let \mathcal{H} denote a stratum of Q_g/Γ , $g > 1$. Then for almost every $q \in \mathcal{H}$,*

$$\limsup_{t \rightarrow \infty} \frac{\log \lambda(h_t \omega)}{\log t} = 1/2.$$

3.2 Diophantine approximation in positive characteristic

In collaboration with Anish Ghosh and Amritanshu Prasad [6, 7], we attempt to study shrinking target properties in positive characteristic by extending dynamical techniques which have been successful in characteristic zero (in particular the methods of [16]).

Let k be a local field of characteristic $p > 2$, and G be the k -points of a semisimple linear algebraic group defined over k , and $\Gamma \subset G$ a non-uniform lattice. Let $X = X_G$ denote the Bruhat-Tits building of G , ∂X the geodesic boundary of X , $Y = X/\Gamma$, and $\pi : X \rightarrow Y$ be the natural projection. Fix a G -invariant metric d on X (we will also use d to denote the metric on the quotient Y). Let $\{g_t(x, \theta)\}_{t \geq 0}$ denote the geodesic starting at the vertex $x \in X$ in direction $\theta \in \partial X$. There is a natural measure class on ∂X .

Theorem 11. [6] *There is a $l = l(Y) > 0$ such that for any $x \in X$, $y \in Y$ and almost all $\theta \in \partial X$,*

$$\limsup_{t \rightarrow \infty} \frac{\log d_Y(\pi(g_t(x, \theta)), y)}{\log t} = 1/l.$$

Here also, it would be interesting to study the behavior of *unipotent* subgroups as well as the recurrence behavior to compact subsets of Y .

A nice application of our results is to metric diophantine approximation in the field $K = \mathcal{F}_q((t^{-1}))$, where $q = p^n$ is a prime power, and \mathcal{F}_q denotes the finite field of order q . A simple example theorem:

Theorem 12. *For almost all $f \in K$, there are infinitely many solutions $P, Q \in k[t]$ to*

$$|f - P/Q| \leq a(|Q|)/|Q|^2$$

if and only if $\int_1^\infty \frac{a(x)}{x} dx$ diverges.

3.3 Flat surface flows and interval exchange maps

As seen from section 2.1, linear flows on flat surfaces (particularly those arising from billiards) are of particular interest to me. To a linear flow on a (singular) flat surface, one can associate an interval exchange map (by taking a first return map of the linear flow to a transverse interval). Recall that an interval exchange map $T_{\lambda, \pi}$, $\lambda \in \Delta_m$, $\pi \in S_m$ is a piecewise isometry of the interval $[0, 1]$ given by splitting $[0, 1]$ into m subintervals of length $\lambda_1, \dots, \lambda_m$, and re-assembling them according to the permutation π (here, $\Delta_m \subset \mathbb{R}_+^m$ is the collection of points of L^1 -norm 1). These are natural generalizations of rotations of the circle. Various shrinking target properties for rotations of the circle were shown by Tseng [24], and in recent joint work with C. Ulcigrai [10], we have shown the following logarithm law:

Theorem 13. *Let $\pi \in S_m$ be an irreducible permutation (that is, $\pi(\{1, \dots, l\}) \neq \{1, \dots, k\}$ for any $k < m$). Then for almost all (with respect to the Lebesgue*

measure class) $\lambda \in \Delta_m$, for all non-singular $x \in [0, 1]$ (that is, $T^n x$ is well-defined for all $n > 0$),

$$\limsup_{n \rightarrow \infty} \frac{-\log T^n x}{\log n} = 1.$$

This is part of continuing work-in-progress, and we hope to build to a deeper understanding of the relation between the shrinking target properties of IET's and flat surface flows and the recurrence properties of the associated Teichmüller geodesic trajectories.

References

- [1] J. S. Athreya, *Quantitative recurrence and large deviations for Teichmüller geodesic flow*, Geometriae Dedicata (2006), v. 119, 121-140.
- [2] J. S. Athreya, *Logarithm laws and shrinking target properties*, preprint. math.DS/0808.0741
- [3] J. S. Athreya, A. Bufetov, A. Eskin, and M. Mirzakhani, *Lattice Point Asymptotics and Volume Growth on Teichmüller space*, preprint. math.DS/0610715v1
- [4] J. S. Athreya, A. Eskin, and A. Zorich, *Rectangular billiards and volumes of spaces of quadratic differentials*, in preparation, 2005.
- [5] J. S. Athreya and G. Forni, *Deviation of ergodic averages for rational polygonal billiards*, Duke Math. J. Volume 144, Number 2 (2008), 285-319.
- [6] J. S. Athreya, A. Ghosh, and A. Prasad, *Borel-Cantelli laws and Diophantine approximation in positive characteristic*, in preparation.
- [7] J. S. Athreya, A. Ghosh, and A. Prasad, *Ultrametric Logarithm Laws I*, accepted to Discrete and Continuous Dynamical Systems.
- [8] J. S. Athreya and G. A. Margulis, *Logarithm laws for unipotent flows*, preprint, submitted to the Journal of Modern Dynamics.
- [9] J. S. Athreya and Y. Minsky, *Logarithm law for horocycle flow on spaces of quadratic differentials*, preprint.
- [10] J. S. Athreya and C. Ulcigrai, *Logarithm laws for interval exchange maps*, in preparation.
- [11] A. Avila, S. Gouezel, and J. -C. Yoccoz, *Exponential mixing for the Teichmüller flow*, Publ. Math. IHES, v. 104, 143-211, 2006.
- [12] A. Eskin and H. Masur, *Asymptotic Formulas on Flat Surfaces*, Ergodic Theory and Dynamical Systems, v.21, 443-478, 2001.

- [13] A. Eskin and A. Okounkov, *Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials*, Invent. Math., v.145, no. 1, 59-103, 2001.
- [14] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Annals of Math., v.155, no. 1, 1-103, 2002.
- [15] S. Kerckhoff, H. Masur, and J. Smillie, *Ergodicity of billiard flows and quadratic differentials*, Annals of Mathematics, v. 124, no. 2, 293-311, 1986.
- [16] D. Y. Kleinbock and G. A. Margulis, *Logarithm laws for flows on homogeneous spaces*, Invent. Math. 138 (1999), no. 3, 451-494.
- [17] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys., v. 147, 1-23, 1992.
- [18] G.A. Margulis, *On some aspects of the theory of Anosov flows*, Ph.D. Thesis, 1970, Springer, 2003.
- [19] H. Masur, *Interval exchange transformations and measured foliations*, Annals of Mathematics, v. 115, 169-200, 1982.
- [20] H. Masur, *Hausdorff dimension of the set of nonergodic foliations of a quadratic differential*, Duke Math. J., v. 66, 387-442, 1992.
- [21] F. Maucourant, *Dynamical Borel-Cantelli lemma for hyperbolic spaces*, Israel J. Math. 152 (2006), 143-155.
- [22] Y. Minsky, and B. Weiss, *Non-divergence of horocyclic flows on moduli space*, J. Reine Angew. Math. v. 552, 131-177, 2002.
- [23] D. Sullivan, *Disjoint spheres, approximation by quadratic numbers and the logarithm law for geodesics*, Acta Mathematica 149 (1982), 215-237.
- [24] J. Tseng, *On circle rotations and the shrinking target properties*, preprint.
- [25] W. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Annals of Mathematics, v. 115, 201-242, 1982.
- [26] A. Zemljakov and A. B. Katok, *Topological transitivity of billiards in polygons*, Mat. Zametki, v. 19, no. 2, 291-300, 1975.