DIFFERENTIABILITY OF LIPSCHITZ MAPS FROM METRIC MEASURE SPACES TO BANACH SPACES WITH THE RADON NIKODYM PROPERTY

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Abstract. We prove the differentiability of Lipschitz maps $X \to V$ where X denotes a PI space i.e. a complete metric measure space satisfying a doubling condition and a Poincaré inequality, and V denotes a Banach space with the Radon-Nikodym Property (RNP). As a consequence, we obtain a bi-Lipschitz nonembedding theorem for RNP targets. The proof of the differentiation theorem depends on a new characterization of the differentiable structure for PI spaces, in terms of directional derivatives in the direction of velocity vectors to rectifiable curves. In particular, we give two proofs that such velocity vectors span the the tangent space at almost every point. The second of these involves a new characterization of the minimal upper gradient. Our structural results have strong implications for the behavior of PI spaces at the infinitesimal level. They will be discussed elsewhere.

1. Introduction

In this paper we will use the term *PI space* to refer to a λ-quasi-convex complete metric measure space (X, d^X, μ) satisfying a doubling condition

$$\mu(B_{2r}(x)) \leq 2^{\kappa} \cdot \mu(B_r(x)) \,, \tag{1.1}$$

and p-Poincaré inequality,

$$\fint_{B_r(x)} |f - f_{x,r}| \, d\mu \leq \tau r \left(\fint_{B_{\lambda r}(x)} g^p \, d\mu \right)^{\frac{1}{p}} \,, \tag{1.2}$$

where $x \in X$, $r \in (0, \infty)$, f is a continuous function, g is an upper gradient for f,

$$\fint_A f \, d\mu := \frac{1}{\mu(A)} \int_A f \, d\mu \,,$$

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$$f_{x,r} := \fint_{B_r(x)} f\, d\mu\,;$$

see [HK96, Che99, Hei01]. We will also assume that the collection of measurable sets is the completion of the Borel σ-algebra with respect to μ (i.e. every subset of a set of measure zero is measurable) [Roy88, p.221]; we impose this assumption because of the appearance of Suslin sets in Sections 3 and 4. Sometimes, by abuse of language, we just say that X is a PI space. The notation above, in particular the space X, the measure μ, as well as the constants κ and λ, will be maintained throughout the paper.

In [Che99], a differentiation theory for real valued Lipschitz functions on PI spaces was given. The notion of differentiation is expressed in terms of an atlas. An atlas consists of a countable collection $\{(U_\alpha, y^\alpha)\}_{\alpha\in\mathcal{A}}$ of *charts*, where the U_α's are measurable subsets, $\mu(X \setminus \bigcup_{\alpha\in\mathcal{A}} U_\alpha) = 0$, $y^\alpha : X \to \mathbf{R}^{k(\alpha)}$ is Lipschitz, and the charts satisfy certain additional conditions. We put $y^\alpha = (y_1^\alpha, \cdots, y_{k(\alpha)}^\alpha)$.

Let V denote a Banach space.

Definition 1.3. A Lipschitz map $f : X \to V$ is *differentiable almost everywhere with respect to the atlas* $\{(U_\alpha, y^\alpha)\}_{\alpha\in\mathcal{A}}$ if there is a collection

$$\left\{ \frac{\partial f}{\partial y_m^\alpha} : U_\alpha \longrightarrow V \right\}_{\alpha\in\mathcal{A},\ 1\le m\le k(\alpha)}$$

of Borel measurable functions uniquely determined μ-almost everywhere, such that for almost every $\underline{x} \in U_\alpha$,

$$f(x) = f(\underline{x}) + \sum_{m=1}^{k(\alpha)} \frac{\partial f}{\partial y_m^\alpha}(\underline{x})(y_m^\alpha(x) - y_m^\alpha(\underline{x})) + o(d^X(x,\underline{x}))\,. \tag{1.4}$$

We will say that f is differentiable at a specific point $\underline{x} \in X$ if (1.4) holds for that point.

The case $V = \mathbf{R}$ of Definition 1.3 was considered in [Che99]; one of the main results there [Che99, Theorem 4.38] was the existence of an atlas with respect to which every Lipschitz function $f : X \to \mathbf{R}$ is differentiable almost everywhere. We will fix such an atlas throughout the paper. It follows readily from the definitions that if (U_α, y^α) and $(\bar{U}_\alpha, \bar{y}^{\bar{\alpha}})$ are charts from two such atlases, then the matrix of partial derivatives $\frac{\partial y_m^\alpha}{\partial \bar{y}_{\bar{m}}^{\bar{\alpha}}}$ is defined and invertible almost everywhere in the overlap $U_\alpha \cap \bar{U}_{\bar{\alpha}}$. This also yields bi-Lipschitz invariant measurable tangent bundle TX.

In this paper we show:

Theorem 1.5. *Every Lipschitz map from X into a Banach space with the Radon-Nikodym Property is differentiable μ-almost everywhere.*

We recall that a Banach space V has the *Radon-Nikodym Property* (RNP) if every Lipschitz map $f : \mathbf{R} \to V$ is differentiable almost everywhere with respect to Lebesgue measure. Since $\mathbf{R}$ is an example of a PI space, Theorem 1.5 is optimal in the sense that the class of Banach space targets considered is maximal.

Just as in [Che99], [CK06b], the differentiation theorem above imposes strong restrictions on PI spaces which bi-Lipschitz embed in RNP targets, and may therefore be used to deduce nonembedding theorems.

Theorem 1.6. *If X admits a bi-Lipschitz embedding in a Banach space with the RNP, then for μ-a.e. $x \in X$, every tangent cone at x is bi-Lipschitz homeomorphic to a Euclidean space.*

From this we obtain:

Corollary 1.7. *The Laakso space* [Laa00] *and the Bourdon-Pajot spaces* [BP99] *do not bi-Lipschitz embed in any RNP Banach space.*

Since by [CK06a] the Laakso space does bi-Lipschitz embed in L^1, we have:

Corollary 1.8. *There exists a doubling metric space which bi-Lipschitz embeds in L^1, but not in any RNP Banach space.*

The proof of Theorem 1.6 is based on a new result which links the differentiation theory on PI spaces with derivatives along curves; see Theorem 3.3. A special case is the following:

Theorem 1.9. *Suppose $f : X \to V$ is a Lipschitz map into an RNP Banach space. Then for a full measure set of points $p \in X$, there is a spanning set $\{v_1, \ldots, v_k\} \subset T_pX$, such that for all $i \in \{1, \ldots, k\}$, the tangent vector v_i is the velocity of a Lipschitz curve $c : I \to X$ at $t \in I$, and $f \circ c$ is differentiable at t; see Section 3 for the definition of the velocity vector of a curve.*

Theorem 3.3 is a contribution to the structure theory of PI spaces of independent interest.

Using some of the same ingredients, in Section 4 we also give a new characterization of the minimal upper gradient of a Lipschitz function on a PI space. We use this give a second proof of Theorem 1.9;

elsewhere, we will use it to give a strengthened version of Theorem 1.9 which leads to fundamental new developments in the infinitesimal structure of PI spaces.

Discussion of the proof.

To prove Theorem 1.5, as in [CK06b] we realize V as an isometrically embedded subspace of an inverse limit space $\underleftarrow{\lim} W_i$, where $\underleftarrow{\lim} W_i$ is the inverse limit of an inverse system of finite dimensional Banach spaces. This allows us to use the differentiability theory for $\mathbf{R}^k$-valued Lipschitz functions to obtain a *weak derivative* for any Lipschitz map $f : X \to V$, which is a map taking values in $\underleftarrow{\lim} W_i$.

The first step in the proof of Theorem 1.5 is to show that if the weak derivative of f takes values in the subspace $V \subset \underleftarrow{\lim} W_i$, then f is differentiable almost everywhere. The argument for this is brief, and combines the inverse limit setup with Lusin's theorem, and the Poincaré inequality.

The remainder of the proof, which appears in Section 3, is devoted to proving that the weak derivative of f takes values in V. The argument for this goes as follows. If $c : I \to X$ is a Lipschitz curve, then the composition $f \circ c$ is differentiable almost everywhere because V has the RNP. Hence the weak derivative of $f \circ c$ coincides with its usual derivative, and in particular, lies in V. When the curve c has a well-defined measurable velocity $c' : I \to TX$ and the chain rule is applicable, for almost every $t \in I$ the weak derivative evaluated on $c'(t)$ will be the same as $(f \circ c)'(t)$, which belongs to V. In this way one reduces the proof to showing that for a full measure set of points $x \in X$, the tangent space T_xX is spanned by the velocities of such curves c. When the PI space X is Euclidean, the above assertion follows readily from Fubini's theorem. In the general case the link between velocities of curves and the tangent space T_xX is much less evident, compare the definition of the tangent bundle after (2.2) and the definition of velocities, Definition 3.1. The development of this connection in Theorem 3.3 constitutes the hard part of the proof, and is of independent interest.

Relation with previous work.

By using an embedding $V \subset \underleftarrow{\lim} W_i$ as described above, a version of Theorem 1.5 was proved in [CK06b] for a class of separable targets with a property that was termed *Good Finite Dimensional Approximation* (GFDA). It was shown in [CK06b] that separable dual spaces have the GFDA property. An essential ingredient in the proof of the differentiation theorem of [CK06b] was to show that if V is a GFDA, then $V = \varprojlim W_i$. It follows trivially that for GFDA targets, the weak derivative lies in V; compare the discussion above. With the observation in [CK08] that inverse limits $\varprojlim W_i$ are just the duals of a separable Banach spaces, it followed that the class of GFDA's is precisely the class of separable dual spaces, a strictly smaller class than that of separable spaces with the RNP; see [MCO80], [Bou81], [Bou83].

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2. Inverse limits and weak derivatives

We begin by briefly recalling some facts from [CK06b], [CK08].

A *standard inverse system* is an inverse system

$$W_1 \xleftarrow{\theta_1} W_2 \xleftarrow{\theta_2} \dots \xleftarrow{\theta_{i-1}} W_i \xleftarrow{\theta_i} \dots , \tag{2.1}$$

where the W_i's are finite dimensional Banach spaces and the bonding maps θ_i are 1-Lipschitz. We recall that by definition, the inverse limit $\varprojlim W_i$ consists of all sequences $(w_1, w_2, \ldots)$, where $w_i \in W_i$, the compatibility condition $\theta_i(w_{i+1}) = w_i$ holds for all I, and $\sup_i \|w_i\| < \infty$. The norm on $\varprojlim W_i$ is defined by $\|(w_1, \ldots)\| = \lim_{i\to\infty} \|w_i\|$. The projection map $\pi_i : \varprojlim W_i \to W_i$ is given by $\pi_i(w_1, \ldots) = w_i$; this is a 1-Lipschitz map. All inverse systems considered in the remainder of the paper will be standard inverse systems.

If V is a separable Banach space, then V may be isometrically embedded in the inverse limit $\varprojlim W_i$ of a standard inverse system [CK06b]. To achieve this, let $\{\phi_i\}_{i\in\mathbb{N}}$ be a norming sequence, i.e. each $\phi_i \in V^*$ has unit norm, and

$$\|v\| = \sup_i \; |\phi_i(v)|$$

for all $v \in V$. For all i, let $V_i := \bigcap_{j \le i} \ker(\phi_j)$. Then we have a decreasing sequence of closed subspaces

$$V \supset V_1 \supset V_2 \supset \dots ,$$

and we define W_i to be the space V/V_i, equipped with the quotient norm. Using the quotient maps $V/V_{i+1} \to V/V_i$ as bonding maps, we obtain a standard inverse system into which V embeds isometrically.

We now return to our PI space X.

Let T^*X denote the measurable cotangent bundle of X, and let $f : X \to V$ denote a Lipschitz map which is differentiable almost everywhere in the sense of Definition 1.3. The differential *df* is the bounded measurable section of $T^*X \otimes V$ whose expression in the canonical trivialization of $T^*X \otimes V$ over U_α is

$$df = \left(\frac{\partial f}{\partial y_1^\alpha}, \dots, \frac{\partial f}{\partial y_{k(\alpha)}^\alpha} \right) ; \tag{2.2}$$

compare (1.4).

Recall that by definition, the tangent bundle TX is the dual bundle of the cotangent bundle T^*X. It will be convenient to work with the derivative $D_x f : TX \to V$, which coincides with the differential *df* under the identification $\mathrm{Hom}(TX, V) \simeq T^*X \otimes V$. Thus,

$$D_x f(z) = \sum_{m=1}^{k(\alpha)} \frac{\partial f}{\partial y_m^\alpha}(x) z_m , \tag{2.3}$$

where $x \in U_\alpha$ and $z = \sum_m z_m \frac{\partial}{\partial y_m^\alpha} \in TX_x$.

Now suppose $V \subset \varprojlim W_i$ realizes V as an isometrically embedded subspace of the inverse limit of a standard inverse system $\{W_i\}$.

If $f : X \to V$ is a Lipschitz (but not a priori differentiable) map, put $f_i = \pi_i \circ f : X \to W_i$. For μ-a.e. $x \in X$, if $v \in T_x X$, the collection of directional derivatives $\{D_x f_i(v)\}$ determines a norm bounded compatible sequence in the inverse system $\{W_i\}$, and we thereby obtain a *weak derivative* $\{D_x f_i\} : TX \to \varprojlim W_i$. The weak derivative is *weakly measurable*, in the sense that its composition with $\pi_j : \varprojlim W_i \to W_j$ is measurable, for all j.

Remark 2.4. When a Lipschitz map $f : X \to V \subset \varprojlim W_i$ is differentiable almost everywhere, the weak derivative is the true derivative, i.e. for μ-a.e. $x \in X$, we have $D_x f = \{D_x f_i\}$. This follows readily from the definitions. In particular, it follows that in this case the weak derivative is a Borel measurable mapping.

Thus far in this section we have not explicitly invoked the Poincaré inequality. The next proposition, which is the converse of the measurability statement in the preceding remark, will make use of it.

Proposition 2.5. *Let $f : X \to V$ be a Lipschitz map, where V is an arbitrary separable Banach space, and suppose the weak derivative $\{Df_i\} : TX \to \varprojlim W_i$ is a Borel measurable mapping, with respect to the measurable vector bundle structure on TX. Then f is differentiable almost everywhere.*

Proof. Since $\{Df_i\} : TX \to \varprojlim W_i$ is measurable, by Lusin's theorem, for almost every $\underline{x} \in X$ there is an $\alpha \in \mathcal{A}$ and a measurable subset $A \subset U_\alpha$, such that $\underline{x} \in A$ is a density point of A, and $\{\frac{\partial f_i}{\partial y^\alpha_m}\} : U_\alpha \to \varprojlim W_i$ is continuous on A for all $m \in \{1, \ldots, k(\alpha)\}$.

Let ℓ denote the function on the right-hand side of (1.4), where the partial derivative $\frac{\partial f}{\partial y^\alpha_m}$ is replaced by

$$\{D_{\underline{x}} f_i\} \left(\frac{\partial}{\partial y^\alpha_m} \right) = \left\{ \frac{\partial f_i}{\partial y^\alpha_m}(\underline{x}) \right\} .$$

Put $\ell_i = \pi_i \circ \ell$. Then $D_x \ell$ is constant in the canonical local trivialization of TX on U_α, and by the assumed continuity of $\{\frac{\partial f_i}{\partial y^\alpha_m}\}$ on A, for all $x \in A$ and all i, we have

$$\lim_{x \to \underline{x}} \|D_x f_i - D_x \ell_i\| = 0 ,$$

where the convergence is uniform in i. Hence, by the Poincaré inequality applied to the function $f_i - \ell_i$, and the fact that f is Lipschitz, the quantity

$$\sup_{x \in B_r(\underline{x})} \frac{\|f_i(x) - \ell_i(x)\|}{r} \tag{2.6}$$

tends to zero as $r \to 0$, uniformly in i. Thus at $\underline{x}$, the weak derivative is a true derivative.

□

In fact, to obtain the measurability required in Proposition 2.5 it suffices for the weak derivative to lie in a separable subspace of $\underleftarrow{\lim}\, W_i$:

Lemma 2.7. *Suppose the weak derivative* $\{D_x f_i\}$ *takes values in a separable subspace* $Y \subset \underleftarrow{\lim}\, W_i$ *for* μ*-a.e.* $x \in X$*. Then* f *is differentiable almost everywhere.*

Proof. By Proposition 2.5, it suffices to show that the weak derivative is Borel measurable.

For each i, let $\{\phi_{i,k}\}_{k\in\mathbb{N}} \subset W_i^*$ be a norming sequence for W_i, i.e. $\|\phi_{i,k}\| = 1$, and for every $w \in W_i$ we have

$$\|w\| = \sup_k\ \phi_{i,k}(w)\,.$$

Then $\{\phi_{i,k} \circ \pi_i\}_{i,k\in\mathbb{N}}$ is a norming family for $\underleftarrow{\lim}\, W_i$ and hence for the separable subspace $Y \subset \underleftarrow{\lim}\, W_i$ as well. By (the argument of) Pettis' theorem, the measurability of the weak derivative follows from the measurability of the compositions $\phi_{i,k} \circ \pi_i \circ f : X \to \mathbf{R}$ for all $i, k \in \mathbb{N}$.

□

3. Velocities of curves

To show that the weak derivative $\{Df_i\} : TX \to \underleftarrow{\lim}\, W_i$ takes values in V, and thereby complete the proof of Theorem 1.5, we will use directional derivatives along rectifiable curves, as indicated in the introduction. To formalize this, we need to make precise the notion of the velocity vector of a curve, as an element of the tangent space. Recall that for PI spaces, the tangent space is defined as the dual of the cotangent space, and hence there is no a priori connection between the tangent space and the as yet to defined velocity vectors of curves.

Velocities and the chain rule.

Let $X_0 \subset X$ be a full μ-measure subset such that for every $x \in X_0$, if $x \in U_\alpha \cap U_\beta$, then x is a point of differentiability of y^α_m with respect to the chart y^β, for all $m \in \{1, \dots, k(\alpha)\}$.

Definition 3.1. If $c : I \to X$ is a Lipschitz curve, $t \in I$ is a point of differentiability of $y^\alpha \circ c$ for all $\alpha \in \mathcal{A}$, and $c(t) \in X_0$, then the *velocity* *of c at t* is defined to be the tangent vector

$$c'(t) = \sum_{m=1}^{k(\alpha)} (y^\alpha_m \circ c)'(t) \frac{\partial}{\partial y^\alpha_m} \in T_{c(t)} X \,.$$

Note that this definition makes sense because of the choice of the set X_0.

With this definition, the chain rule becomes:

Lemma 3.2. *Suppose $c : I \to X$ is a Lipschitz curve, and the velocity vector $c'(t) \in TX$ is defined.*

1. For any Lipschitz function $u : X \to \mathbf{R}$ which is differentiable with respect to the atlas $\{(U_\alpha, y^\alpha)\}$ at x, the derivative $(u \circ c)'(t)$ is defined, and

$$(u \circ c)'(t) = (D_{c(t)} u)(c'(t)) \,.$$

2. If $c(t) \in X$ is a point of weak differentiability of $f : X \to V$ (i.e. x is a point of differentiability of f_i for all i), and t is a point of differentiability of $f \circ c$, then we have the following chain rule relating the derivative of $f \circ c$ and the weak derivative of f:

$$(f \circ c)'(t) = \{(f_i \circ c)'(t)\} = \{(D_{c(t)} f_i)(c'(t))\} \,.$$

In particular, the weak derivative of f in the direction of the velocity vector $c'(t)$ lies in $V \subset \varprojlim W_i$.

The proof is straightforward.

Velocities span the tangent space.

Next we prove the following:

Theorem 3.3. *Fix a countable collection $\Phi = \{\phi_i : X \to V_i\}$ of Lipschitz maps into RNP Banach spaces. Let $\mathcal{V}_0$ be the collection of tangent vectors $v \in TX$ such that there is a 1-Lipschitz curve $c : I \to X$ and $t \in I$, where $v = c'(t)$, and t is a point of differentiability of $\phi_i \circ c$ for all i. Then there is a set $Z \subset X$ with $\mu(X \setminus Z) = 0$, such that $\mathcal{V}_0 \cap T_z X$ spans the fiber $T_z X$ at every point $z \in Z$.*

Remark 3.4. Elsewhere we will show that for a full measure set of $x \in X$, there is a dense set of directions in $\mathcal{V}_0 \cap T_x X$, i.e. the set of rays in $T_x X$ which intersect $\mathcal{V}_0$ is dense in $T_x X$. However, we will not need this finer result here.

Proof of Theorem 1.5 using Lemma 2.7 and Theorem 3.3. Applying Theorem 3.3 with $\Phi = \{f : X \to V\}$, we obtain a full measure subset $Z \subset X$ as in the Theorem. Let $W \subset Z$ be a full measure subset where f is weakly differentiable. Then for every $x \in W$, and every $v \in \mathcal{V}_0 \cap T_xX$, we have $\{(D_xf_i)(v)\} \in V$, by part 2 of Lemma 3.2. Since $\mathcal{V}_0 \cap T_xX$ spans T_xX, it follows that $\{(D_xf_i)(T_xX)\} \subset V$. Hence f is differentiable almost everywhere by Lemma 2.7.

We now turn to the proof of Theorem 3.3.

Let $\mathcal{V}$ be the (fiberwise) span of $\mathcal{V}_0$, i.e. $\mathcal{V} \cap T_xX = \operatorname{span}(\mathcal{V}_0 \cap T_xX)$.

We begin with a preview of the argument. We will first show that $\mathcal{V}$ defines a measurable sub-bundle of TX. If there is a positive measure set of points $x \in X$ where $\dim(\mathcal{V} \cap T_xX) < \dim T_xX$, then by Lusin's theorem, we may pass to a positive measure subset A of some U_α with the same property, where in addition $\mathcal{V}$ lies in a continuous codimension 1 sub-bundle E of the $k(\alpha)$-dimensional bundle $TX\Big|_A$, where the continuity is defined with respect to the bundle chart induced by y^α. If p is a density point of A, and u is a linear combination of coordinates $y_1^\alpha, \ldots, y_{k(\alpha)}^\alpha$ whose derivative at p has kernel $E \cap T_pX$, then one finds that there is an upper gradient ρ for u such that

$$\lim_{\substack{x \to p, \\ x \in A}} \rho(x) = 0.$$

This implies that the average of ρ over $B_r(p)$ tends to zero as $r \to 0$, which contradicts the nondegeneracy of u.

We now give the details. Our first step is:

Lemma 3.5. *The sub-bundle $\mathcal{V} \subset TX$ is measurable.*

Prior to proving the lemma, we recall some facts about Suslin sets [Fed69]. A subset of a metric space is *Suslin* if it is the continuous image of a Borel subset of a complete separable metric space. Suslin sets in a complete, σ-finite Borel regular measure space such as X, are μ-measurable; this is why we assumed that the measure μ is complete. Note that if Z is a complete separable metric space, then the image of a Suslin set $S \subset Z$ under a Suslin measurable mapping $\tau : Z \to X$ is also Suslin (because the graph of τ is a Suslin subset of $Z \times X$).

Proof. In brief, the proof is a straightforward application of the facts about Suslin sets recalled above.

It suffices to show that for each α, the restriction of $\mathcal{V}$ to U_α is measurable. We may assume that U_α is Borel measurable, and that it is contained in the set X_0 defined before Definition 3.1.

Let Γ denote the space of 1-Lipschitz maps $c : [0,1] \to X$, equipped with the compact-open topology. Then Γ is a complete, separable metric space, since X is complete and doubling. Consider the collection $S_0 \subset \Gamma \times [0,1]$ of pairs (c,t) such that $c(t) \in U_\alpha$, and the composition $\phi_i \circ c : [0,1] \to V_i$ is differentiable at t for all i. This is easily seen to be a Borel set. Also, the map $\sigma : S_0 \to \mathbf{R}^{k(\alpha)}$ which sends $(c,t) \in S_0$ to

$$(D_{c(t)}y^\alpha)(c'(t)) = (y^\alpha \circ c)'(t) \in \mathbf{R}^{k(\alpha)}$$

is Borel measurable.

For $j \in \{1,\ldots,k(\alpha)\}$, let T_j be the set of points $x \in U_\alpha$ where the fiber $\mathcal{V} \cap T_xX$ has dimension j. We claim that T_j is a Suslin subset of X. To see this, let S_1 be the set of j-tuples $((c_1,t_1),\ldots,(c_j,t_j)) \in S_0^j$ such that $c_1(t_1) = \ldots = c_j(t_j)$; this is a closed subset of S_0^j. Then let $S_2 \subset U_\alpha \times \wedge^j \mathbf{R}^{k(\alpha)}$ be the image of S_1 under the Borel map $S_1 \to U_\alpha \times \wedge^j \mathbf{R}^{k(\alpha)}$ which sends $((c_1,t_1),\ldots,(c_j,t_j))$ to

$$(c_1(t_1), \sigma((c_1,t_1)) \wedge \ldots \wedge \sigma((c_j,t_j)) \in U_\alpha \times \wedge^j \mathbf{R}^{k(\alpha)}\,.$$

Then $\cup_{k \geq j} T_k$ – the set of points where the fiber of $\mathcal{V}$ has dimension at least j – is the projection of $S_2 \cap (U_\alpha \times (\wedge^j \mathbf{R}^{k(\alpha)} \setminus \{0\}))$ to U_α, and is therefore a Suslin set and μ-measurable. It follows that T_j is μ-measurable for all $j \in \{1,\ldots,k(\alpha)\}$.

Let $\bar{T}_j \subset T_j$ be a full measure Borel subset of T_j. Then $\cup_{j=1}^{k(\alpha)} \bar{T}_j$ has full measure in U_α.

Fix $j \in \{1,\ldots,k(\alpha)\}$. Let $G(j,k(\alpha))$ denote the Grassman manifold of j-planes in $\mathbf{R}^{k(\alpha)}$. Then there is a well-defined map $\gamma_j : \bar{T}_j \to G(j,k(\alpha))$ such that for every $x \in \bar{T}_j$, the fiber $\mathcal{V} \cap T_xX$ maps under $D_xy^\alpha : T_xX \to \mathbf{R}^{k(\alpha)}$ to the subspace $\gamma_j(x)$ of $\mathbf{R}^{k(\alpha)}$. To see that γ_j is measurable, pick an open subset of $G(j,k(\alpha))$, and observe that its inverse image in $\bar{T}_j$ is a Suslin set, using a construction similar to the one above.

□

We now return to the proof of Theorem 3.3.

Suppose $\dim(\mathcal{V} \cap T_xX)$ is strictly smaller than $\dim T_xX$ for a positive measure set of points $x \in X$. Then for some index $\alpha \in \mathcal{A}$, there is a measurable subset $A \subset U_\alpha$ with $\mu(A) \in (0,\infty)$, where the strict inequality $\dim(\mathcal{V} \cap T_xX) < k(\alpha)$ holds. By Lusin's theorem, we may assume without loss of generality that $\mathcal{V}|A$ is contained in a codimension 1 sub-bundle E of $TX|A$, where E is a continuous sub-bundle relative to the bundle charts given by y^α, i.e. the fiber of E at $x \in A$ is the kernel of $\psi(x) \circ D_x y^\alpha : T_xX \to \mathbf{R}$, for some continuous map $\psi : A \to (\mathbf{R}^{k(\alpha))^*} \setminus \{0\}$. It follows from the defining property of our atlas $\{(U_\alpha, y^\alpha)\}$ – specifically the almost everywhere uniqueness of coefficients appearing in (1.4) – that we may also assume that for every $p \in A$, every nontrivial linear combination of the coordinate functions y^α_m has nonzero pointwise upper Lipschitz constant at p.

Choose $p \in A$, and put $\bar\psi = \psi(p)$.

Lemma 3.6. *There is a continuous function* $\zeta : A \to [0,\infty)$ *such that* $\zeta(x) \to 0$ *as* $x \to p$, *and*

$$|D_x(\bar\psi \circ y^\alpha)(v)| \le \zeta(x),$$

for every $v \in \mathcal{V}_0 \cap T_xX$.

Proof. Let c and t be as in the definition of $\mathcal{V}_0$, so that $c'(t) = v$. Then $D_x(\bar\psi \circ y^\alpha)(v) = \bar\psi((D_x y^\alpha)(v))$. Also, the vector $(D_x y^\alpha)(v)$ has uniformly bounded norm since y^α is Lipschitz, and it lies in the hyperplane $\ker \psi(x) \subset \mathbf{R}^{k(\alpha)}$, which approaches $\ker \bar\psi$ as $x \to p$. The lemma follows. ☐

Define a function $\rho : X \to [0,\infty)$ by $\rho(x) = \zeta(x)$ if $x \in A$, and $\rho(x) = L$ otherwise, where L is the Lipschitz constant of $\bar\psi \circ y^\alpha$. We claim that ρ is an upper gradient for $\bar\psi \circ y^\alpha$. To see this, we need only show that if $c : I \to X$ is a 1-Lipschitz curve, then for almost every $t \in I$, we have

$$|\bar\psi \circ y^\alpha \circ c)'(t)| \le \rho \circ c(t)\,.$$

If $t \in I$ is such that $c(t) \notin A$, then this obviously holds, since $\bar\psi \circ y^\alpha \circ c$ is L-Lipschitz. If $t \in I$ is a point such that $c(t) \in A$ and the derivatives $(y^\alpha \circ c)'(t)$ and $(\phi_i \circ c)'(t)$ are defined for all i, then $c'(t)$ is defined and lies in $\mathcal{V}_0$. Therefore the chain rule applies, and

$$|(\bar\psi \circ y^\alpha \circ c)'(t)| = |\bar\psi((D_{c(t)} y^\alpha)(c'(t))| \le \zeta(c(t)) = \rho \circ c(t)$$

by Lemma 3.6. The remaining points $t \in I$ have measure zero.

Now let $p \in A$ be a density point of A. Applying the Poincaré inequality to $\psi \circ y^\alpha$ on balls $B(p,r)$, using the fact that p is a density point of Z and $\psi \circ y^\alpha$ is Lipschitz, we conclude the pointwise upper Lipschitz constant of $\psi \circ y^\alpha$ at p is zero:

$$\limsup_{r\to 0} \frac{\sup\ \{\psi \circ y^\alpha(x) - \psi \circ y^\alpha(p) \mid x \in B(p,r)\}}{r} = 0.$$

This is a contradiction to the choice of A, which completes the proof of the theorem.

$\square$

4. A NEW CHARACTERIZATION OF THE MINIMAL UPPER GRADIENT

In this section will give a new characterization of the minimal upper gradient. We then apply this to give a different proof of Theorem 3.3. It will also play a role in the proof of the stronger version of Theorem 3.3 alluded to in Remark 3.4.

Generalized upper gradients and minimal upper gradients.

Recall that if $u : X \to \mathbf{R}$ is a Lipschitz function, then a Borel measurable function $g : X \to [0,\infty]$ is a *generalized upper gradient* if there is a sequence of Lipschitz functions $u_k : X \to \mathbf{R}$, and a sequence $g_k \in L^p_{\mathrm{loc}}(X)$ such that g_k is an upper gradient for u_k for all k, and $u_k \overset{L^p_{\mathrm{loc}}}{\to} u$, $g_k \overset{L^p_{\mathrm{loc}}}{\to} g$ [Che99, Section 2]. This is equivalent to being a *p-weak upper gradient*, i.e. satisfing the usual upper gradient condition for all but a set of curves of zero p-modulus [Sha00, Che99]. It was shown in [Che99] that every Lipschitz function $u : X \to \mathbf{R}$ has a *minimal upper gradient*, which is a generalized upper gradient $g_f : X \to \mathbf{R}$ with the property that every generalized upper gradient g satisfies $g \geq g_f$ almost everywhere.

Negligible sets.

Let $\mathcal{C}$ denote the space of 1-Lipchitz curves $c : [0,1] \to X$. With the metric $d(c_1, c_2) = \max_{t\in[0,1]}\ d^X(c_1(t), c_2(t))$, the space $\mathcal{C}$ is complete and separable, as is $\mathcal{C} \times [0,1]$ equipped with the product metric.

Definition 4.1. A subset $N \subset \mathcal{C} \times [0,1]$ will be called *negligible* if it is Borel, and for all $c \in \mathcal{C}$,

$$\mathcal{L}(N \cap (\{c\} \times [0,1])) = 0\,,$$

where $\mathcal{L}$ denotes Lebesgue measure on $[0,1]$. Clearly, a countable union of negligible sets is negligible.

Theorem 4.2. *Let $f : X \to \mathbf{R}$ be a Lipschitz function with minimal upper gradient g_f, and let $N \subset \mathcal{C} \times [0,1]$ be a negligible set. Define a function $\hat{g}_f : X \to [0,\infty)$ by letting $\hat{g}_f(x)$ be the supremum of the set*

$$\{\ |(f \circ c)'(t)| \ \mid \ (c,t) \in \mathcal{C} \times [0,1] \setminus N,\ \ c(t) = x,\ \text{and}\ (f \circ c)'(t)\ \text{exists}\}$$

if it is nonempty, and 0 otherwise. Then $\hat{g}_f$ coincides with g_f almost everywhere.

Proof. We begin by showing that the function $\hat{g}_f$ defined above is μ-measurable. For this it suffices to show that $\hat{g}_f^{-1}((a,\infty))$ is a Suslin set for all $a \in [0,\infty)$.

Let $\pi : \mathcal{C} \times [0,1] \to X$ be the map $\pi((c,t)) = c(t)$. Then π is continuous, and $\hat{g}_f^{-1}((a,\infty))$ is the image under π of the Borel set

$$\{(c,t) \in \mathcal{C} \times [0,1] \setminus N \ \mid \ (f \circ c)'(t)\ \text{exists},\ |(f \circ c)'(t)| \in (a,\infty)\}\ ;$$

hence $\hat{g}_f^{-1}((a,\infty))$ is Suslin, as claimed.

Let $g : X \to \mathbf{R}$ be a Borel measurable function such that $g \geq \hat{g}_f$, and $g = \hat{g}_f$ almost everywhere. Notice that g is an upper gradient for f, because if $c \in \mathcal{C}$, then for a.e. $t \in [0,1]$, the derivative $(f \circ c)'(t)$ exists, $(c,t) \notin N$, and

$$|(f \circ c)'(t)| \leq \hat{g}_f \circ c(t) \leq g \circ c(t)\,.$$

Therefore $\hat{g}_f \geq g_f$ almost everywhere, since g_f is a minimal upper gradient.

Observe that $\hat{g}_f \leq \operatorname{Lip} f$ everywhere; this follows from the fact that if $c \in \mathcal{C}$ and $(f \circ c)'(t)$ exists, then $\operatorname{Lip}_{c(t)}(f) \geq |(f \circ c)'(t)|$ because c is 1-Lipschitz. By [Che99, Thm 6.1], we have $g_f = \operatorname{Lip} f$ almost everywhere, and hence $\hat{g}_f \leq g_f$ almost everywhere. Thus $\hat{g}_f = g_f$ almost everywhere. ☐

Remark 4.3. The full strength of Theorem 4.2 is not used in the application given below. It would be sufficient to know that $g \geq C\, g_f$ μ-a.e., for some $C \in (0,\infty)$ which does not depend on f, and this is considerably easier to prove, see [Che99, Prop. 4.26].

An alternate proof of Theorem 3.3.

Returning to the setting of Theorem 3.3, let $N \subset \mathcal{C} \times [0,1]$ be the negligible set of pairs (c,t) such that one of the compositions $\{y^\alpha \circ c\}_{\alpha \in \mathcal{A}}$ or $\{\phi_i \circ c\}_{\phi_i \in \Phi}$ is not differentiable at t.

For each $\alpha \in \mathcal{A}$, and each rational $k(\alpha)$-tuple $(a_1, \dots, a_{k(\alpha)}) \in \mathbb{Q}^{k(\alpha)}$, let $h_{(a_1,\dots,a_{k(\alpha)})} : X \to \mathbf{R}$ be the function $\hat{g}_f$ defined as in Theorem 4.2, with $f = \sum_m \, a_m \, y^\alpha_m$.

Now let $Z \subset X$ be a full measure Borel set such that:

(1) $Z \subset X_0$, where $X_0 \subset X$ is the set defined before Definition 3.1.
(2) For all $\alpha \in \mathcal{A}$, $(a_1, \dots, a_{k(\alpha)}) \in \mathbb{Q}^{k(\alpha)}$, and $x \in Z$, the function $h_{(a_1,\dots,a_{k(\alpha)})}(x)$ coincides with the pointwise upper Lipschitz constant of the function $f = \sum_m \, a_m \, y^\alpha_m$ at every point in Z.
(3) The function $h_{(a_1,\dots,a_{k(\alpha)})}$ is approximately continuous at every point in Z.

Suppose that the $\mathcal{V}_0 \cap T_x X$ does not span the fiber $T_x X$ at some point $x \in U_\alpha \cap Z$. Then there is a nonzero $k(\alpha)$-tuple $(b_1, \dots, b_{k(\alpha)}) \in \mathbf{R}^{k(\alpha)}$ such that $\sum_m b_m \, (D_x y^\alpha_m)(v) = 0$ for all $v \in \mathcal{V}_0 \cap T_x X$. The chain rule (Lemma 3.2) implies that $\hat{g}_u(x) = 0$, where $u := \sum_m \, b_m \, y^\alpha_m$. If $\{a^j\} \subset \mathbb{Q}^{k(\alpha)}$ is a sequence converging to $b = (b_1, \dots, b_{k(\alpha)})$, then $h_{a^j}(x) \to 0$. Therefore the upper pointwise Lipschitz constant of $\sum_m \, a^j_m \, y^\alpha_m$ at x tends to zero as well. Hence the nontrivial linear combination $\sum_m \, b_m \, y^\alpha_m$ has zero pointwise upper Lipschitz constant at x, which contradicts the fact that the differentials of the y^α_m's are independent on $U_\alpha \cap Z$.

□

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